ON SOME TWO-SAMPLE AND K-SAMPLE RANK TESTS WITH APPLICATIONS TO LIFE TESTING*

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TABLE OF CONTENTS

Chapter I

INTRODUCTION AND SUMMARY

1. General Introduction

During the last twelve years the mathematical theory of life testing and reliability has made rapid strides as a result of the demands of modern technology and as a result of the experiences with complex military systems in World War II. Because of the magnitude of the expenditures involved and the pressure to make early decision it has been necessary to devise suitable statistical techniques that would enable the experimenter to make valid statistical decisions even before the experiment is terminated.

In life testing experiments where several items (say, N) are tested simultaneously to study their expected failure times, it becomes necessary to find suitable methods based on the first few (say, r) observations which are naturally ordered. Any test of the above form will be termed an r out of N test.

The pioneering work in problems of life testing was done by Epstein and Sobel in a series of important and extremely influential papers [10] , [11] , where they assumed that the failure times follow the exponential distribution. The exponential distribution as a model in life testing was readily accepted by many primarily because of the analytical simplicity of the model and because of the simple physical interpretations that can be attached to it.

However, in recent years the universal application of the exponential distribution in life testing experiments has been seriously questioned and various other statistical distributions are being proposed as alternative models. Thus, in the study of life characteristics of electron tubes Kao [20] has proposed the model of the

Weibull distribution which has become very popular in recent industrial applications.

In view of the availability of various alternative models a natural question arises as to how robust the exponential life testing procedure is. The above problem has been attacked, among others, by Barlow and Proschan [2] , Basu [3] , [4] and Zelen and Dannemiller [34] . In [3] Basu has shown the nonrobustness of exponential life test procedures when in reality the true model is a truncated exponential distribution. Zelen and Dannemiller have studied the robustness of exponential life test procedures when the true distribution is Weibull and showed that the exponential life testing procedure is non-robust which has been further confirmed in [4] . Similarly Barlow and Proschan have also shown the non-robustness of exponential life testing procedures when the true model is a distribution with monotone failure rate.

A natural problem is then to develop any robust life testing procedure if possible. Since the distribution-free methods are usually robust the emphasis of statisticians naturally shifts towards developing suitable nonparametric life-testing procedures. Some efforts in this direction have already been made by Epstein [8] , [9] , Eilbott and Nadler [7], and Sobel [28], [29], among others. But essentially all the studies have been confined to two-sample problems only.

2. Problems Considered

The object of this report is to emphasize the importance of non-parametric rank tests that can be adopted to life testing situations. To this end we make a comparative study of the various two-sample r out of N tests already available and then to propose for the first time some k-sample r out of N tests. The basic assumption made throughout has been that we have k-samples ($k \geq 2$) from k-populations with continuous cumulative distribution functions $F_1, F_2, \cdots, F_k$ where the F_i's belong to a family of distribution functions $\mathcal{F}$ indexed by a parameter θ which is either a location or a scale parameter; in the latter case θ is of course assumed positive. Below we give a brief description of our findings which have been reported in greater details in the main body of the thesis.

The thesis is divided into two parts. Part I consists of chapter II only and deals with the various two-sample r out of N tests. We have primarily studied the large sample properties of the statistic $V_r^{m,n}$ proposed by Sobel. The statistic is shown to be consistent and asymptotically normally distributed in the non-null case. A general expression for its efficacy has been derived for both location and scalar alternatives. Finally the performance of the statistic has been compared with various other two-sample r out of N tests.

The two chapters of part II deal with the various possible k-sample extensions of the statistic $V_r^{m,n}$. In chapter III we have proposed the statistic $B_r^{(N)}$ which may be considered as a generalization of both the Kruskal statistic H [21] and a statistic studied by Terpstra [30] . The exact mean and variance of $B_r^{(N)}$ under the null hypothesis have been obtained and finally the asymptotic distribution of $B_r^{(N)}$ both under the null and the non-null case has been found.

Chapter IV deals with two other k-sample extensions of the $V_r^{m,n}$ statistic which are found useful in ranking and selection problems. The first one, the $V(N,r)$ statistic, is shown to be asymptotically normal both under the null and the non-null hypotheses. The mean and variance of $V(N,r)$ under the null hypothesis are found. The possible extreme values of $V(N,r)$ is also investigated.

The second generalization $W(N,r)$ has been studied conditioned under a given observed "pattern". This statistic is also shown to be a generalizations of the statistics proposed by Jonckheere [19] and Terpstra [31] .

Throughout the report our emphasis has been on the applicability of above tests in problems of life testing; however, there are many other situations where the statistics studied in the present report may be found useful. As an example, in bio-assay problems we might be interested in comparing the efficacies of two drugs but we may not afford to wait indefinitely until all the observations are available.

3. Note on Notation

Finally, it seems necessary to explain the numbering system we have used for various sections and equations in the report. Unless otherwise mentioned, by equation (x,y) we shall refer to the equation number y in section x of the same chapter. Similarly we shall number the theorems, lemmas etc. In situations where we had to refer to equations of a different chapter we have also mentioned the chapter number explicitly. The sections in various chapters have all been numbered starting serially from 1.

Part I

The two-sample case

CHAPTER II

ON THE LARGE SAMPLE PROPERTIES OF A GENERALIZED WILCOXON-MANN-WHITNEY STATISTIC

1. Introduction.

Let $X_1, X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots, Y_n$ be two independent samples of sizes m and n from two populations with continuous cumulative distribution functions (cdf's) F(x) and G(y), where F and G belong to the same family $\mathcal{F}$ of distribution functions indexed by a parameter θ. We wish to test the hypothesis

$$(1.1) \qquad H_0: F = G$$

against the alternative that they are different.

Let all the $m+n = N$ observations be ordered in a sequence and suppose we want to base a decision on (at most) the first r of the combined set of N observations, i.e. we have a right-censored sample of size at most r. For the above problem Sobel [28] has proposed a statistic $V_r^{m,n}$ which we now introduce. Let m_i and n_i be the number of x and y failures, respectively, among the first i ordered observations of the combined sample, so that

$$(1.2) \qquad m_i + n_i = i, \quad i = 1,2,\cdots,r \quad .$$

These observations (x's and y's) are the failure times in a life testing experiment. The proposed statistic is given by

$$(1.3) \qquad V_r^{m,n} \equiv V_r^{(N)} = \sum_{i=1}^{r} (nm_i - mn_i) \quad .$$

In [29] this statistic $V_r^{(N)}$ is shown to be related to the well-known Wilcoxon-Mann-Whitney statistic [22], [32] and the small sample properties of $V_r^{(N)}$ and its exact and asymptotic distribution under the null hypothesis are also discussed.

In view of the usefulness of the above statistic it seems desirable to explore the properties of this statistic further; the object of this chapter is to establish some large-sample properties of the statistic $V_r^{(N)}$. In section 3 we prove the asymptotic normality in the null and non-null case of a statistic (defined in section 2) which is equivalent to $V_r^{(N)}$. Consistency of the test statistic is established in section 4. In section 5 general expressions for the efficacy of the test are given and in section 6 we derive the asymptotic relative efficiency (ARE) of the above test with respect to the likelihood ratio test for testing the scale parameter in the case of the exponential distribution. The performance of the test has been compared with the precedence test in section 7 and with other asymptotically most powerful rank tests for censored sampling in section 8.

Some studies in this direction have recently been made by Halperin [18] and Gehan [15]. However, they consider censoring schemes in which the experiment is terminated after a given period so that r, the number of uncensored observations, becomes a random variable.

2. Relation of $V_r^{(N)}$ to other statistics.

To facilitate discussion we shall first define a new sequence $\{z_i\}$ $(i = 1,2,\cdots,N)$ derived from the combined ordered sample, always counting ordered observations from the left, as follows:

$$(2.1) \qquad z_i = \begin{cases} 1 & \text{if the } i^{th} \text{ ordered observation is an } X \\ 0 & \text{otherwise} \end{cases} .$$

Also let

(2.2)
$$\delta_x^*(i) = \begin{cases} i & \text{if the } (r+1-i)^{th} \text{ ordered observation is an X} \\ C & \text{otherwise,} \end{cases}$$

$$\delta_y^*(i) = \begin{cases} i & \text{if the } (r+1-i)^{h} \text{ ordered observation is a Y} \\ 0 & \text{otherwise } (i=1,2,\cdots,r) \end{cases} .$$

Clearly

(2.3) $$\delta_x^*(r+1-i) = (r+1-i)z_i; \quad \delta_y^*(r+1-i) = (r+1-i)(1-z_i)$$

and

(2.4) $$\delta_x^*(i) + \delta_y^*(i) = i \qquad (i = 1,2,\cdots,r) .$$

We now prove

Lemma 2.1: For any $r \geq 1$

(2.5)
(a) $$\sum_{i=1}^{r} m_i = \sum_{i=1}^{r} \delta_x^*(i) ,$$

(b) $$\sum_{i=1}^{r} n_i = \sum_{i=1}^{r} \delta_y^*(i) .$$

Proof. Note that if the first observation is an x, it will contribute unity to each of m_i's $(i=1,2,\cdots,r)$, whereas it contributes r to $\delta_x^*(r)$ only. In general, if the $(r+1-i)^{th}$ ordered observation is an x it contributes 1 to each of the last i terms $m_{r-i+1}, m_{r-i+2},\cdots,m_r$ on the left side of (2.5a), whereas it contributes i to the right side of (2.5a). On the other hand, if the $(r+1-i)^{th}$ observation is a y it contributes nothing to either side of (2.5a), $(i=1,2,\cdots,r)$. This proves (2.5a) and equation (2.5b) is proved similarly.

Let us define the statistic $T_r^{(N)}$ by

(2.6) $$T_r^{(N)} = \sum_{i=1}^{r} \left(\frac{i-r-1}{N}\right)z_i + \frac{m(r+1)^2}{2N^2}$$
$$= \sum_{i=1}^{N} e_i z_i ,$$

where

$$(2.7)\qquad e_i = \begin{cases} \dfrac{i-r-1}{N} + \dfrac{(r+1)^2}{2N^2} & \text{if } 1 \le i \le r \\ (r+1)^2/2N^2 & \text{if } r < i \le N \end{cases} .$$

The equivalence of $V_r^{(N)}$ and $T_r^{(N)}$ is shown in the following

Theorem 2.1. For testing H_0: $F = G$ against one-sided (or two-sided) alternatives H_1: $F \neq G$, the statistics $V_r^{(N)}$ and $T_r^{(N)}$ are equivalent.

Proof. Using (1.3), (2.4) and (2.5)

$$(2.8)\qquad \begin{aligned} V_r^{(N)} &= n \sum_{i=1}^{r} m_i - m \sum_{i=1}^{r} n_i \\ &= n \sum_{i=1}^{r} \delta_x^*(i) - m \sum_{i=1}^{r} \delta_y^*(i) \quad \text{(by Lemma 2.1)} \\ &= N \sum_{i=1}^{r} \delta_x^*(i) - \frac{mr(r+1)}{2} \\ &= N \sum_{i=1}^{r} (r+1-i)z_i - \frac{mr(r+1)}{2} \\ &= \frac{m(r+1)}{2} - N^2 T_r^{(N)} . \end{aligned}$$

The relationship of $T_r^{(N)}$ (and hence that of $V_r^{(N)}$) with the Wilcoxon statistic W and the Mann-Whitney statistic U becomes clear by putting r=N in (2.6), (that is, when the complete combined sample is available).

$$(2.9)\qquad \begin{aligned} NT_N^{(N)} &= \sum_{i=1}^{N} iz_i - m(N+1) + \frac{m(N+1)^2}{2N} \\ &= W - m(N^2-1)/2N \\ &= U + \frac{m(m+1)}{2} - \frac{m(N^2-1)}{2N} , \end{aligned}$$

where $W = \sum_{i=1}^{N} iz_i$ is the Wilcoxon statistic [32] and U is the Mann-Whitney statistic [22] defined for a sequence of m x's and n y's as the number of y's preceding each x_i, summed from i=1 to m.

3. Asymptotic Normality of $T_r^{(N)}$

The asymptotic normality of $T_r^{(N)}$ can be derived from a theorem (and its corollaries) of Hájek [17]. We first define his notation and state the theorem in a form suitable for our purpose and then show how the theorem applies to $T_r^{(N)}$.

Consider an infinite sequence of random vectors $(X_{N1}, X_{N2}, \cdots, X_{Nv_N})$, $(N=1,2,\cdots)$ of increasing sizes $(v_1 < v_2 < \cdots)$ where $v_N = m_N + n_N$ and all the X_{Ni} (in different vectors and in the same vector) are independent. The cdf of X_{Ni} is given by

$$(3.1) \qquad P(X_{Ni} \leq x) = H(x - \beta c_{Ni}), \quad 1 \leq i \leq v_N, \quad N=1,2,\cdots,$$

where H is a known distribution function, β is the parameter under test (i.e., $\beta = 0$ corresponds to H_0) and the c_{Ni} are defined in terms of known constants by

$$(3.2) \qquad c_{Ni} = \begin{cases} 0 & 1 \leq i \leq m_N \\ \Delta_N & m_N+1 \leq i \leq v_N \end{cases} .$$

Let us make the following assumptions.

<u>Assumption 3.1</u>. The density $h(x) = H'(x)$ exists and $\sqrt{h(x)}$ possesses a square integrable derivative, that is,

$$(3.3) \qquad \int_{-\infty}^{\infty} [h'(x)/h(x)]^2 \, h(x)dx < \infty ;$$

here we are using $f'(x)$ to denote the derivative of $f(x)$.

<u>Assumption 3.2</u>.

$$(3.4) \qquad \lim_{N \to \infty} \max \{m_N^{-1} - v_N^{-1}, n_N^{-1} - v_N^{-1}\} = 0 .$$

$$(3.5) \qquad \sup_N m_N n_N \Delta_N^2 / v_N < \infty .$$

Denoting the inverse function of H by H^{-1}, let us introduce the following notation:

$$(3.6) \qquad \varphi(u) = -\,[h'(H^{-1}(u))/h(H^{-1}(u))] \quad , \qquad 0<u<1 \quad .$$

Let R_{Ni} be the rank of X_{Ni} in the ordered sample; that is, if $X_{N,(i)}$ denotes the i^{th} smallest of the X_{Ni} values so that

$$X_{N,(1)} < \cdots\cdots < X_{N(v_N)} \qquad ,$$

then

$$(3.7) \qquad X_{Ni} = X_{N,(R_{Ni})} \;, \quad 1 \leq i \leq v_N, \quad N=1,2,\cdots \quad .$$

Let $\varphi_N(u)$ be a step function defined on (0,1) and having the property that

$$(3.8) \qquad \varphi_N(u) = \varphi_N\left(\frac{i}{v_N+1}\right), \quad \frac{i-1}{v_N} < u \leq \frac{i}{v_N} \quad .$$

This function will be further specified later. We define the rank-order statistic

$$(3.9) \qquad T_N = \sum_{i=1}^{v_N} (c_{Ni}-\bar{c}_N)\, \varphi_N\left(\frac{R_{Ni}}{v_N+1}\right), \quad N=1,2,\cdots$$

where $\bar{c}_N = \sum_{i=1}^{v_N} c_{Ni}/v_N$. Letting $\mathcal{H}$ denote the family of cdf's satisfying assumption 3.1, we have then the following theorem.

<u>Theorem 3.1</u> (Hájek). Consider model (3.1) and suppose assumptions 3.1 and 3.2 are satisfied. If there exists a distribution $H \in \mathcal{H}$ such that

$$(3.10) \qquad \lim_{N \to \infty} \int_0^1 [\varphi_N(u) - \varphi(u)]^2 \, du = 0$$

where $\varphi = \varphi(H)$ is defined by (3.6), then T_N is asymptotically normally distributed not only for this H but for any distribution in $\mathcal{H}$.

To see how the theorem applies in our case let us first note that in our case $(X_{N1}, X_{N2}, \cdots, X_{Nv_N})$ denotes the sequence consisting of $m_N(=m)$ X's and $n_N(=n)$ Y's and $v_N=N$. (We have attached the subscript N to indicate the dependence of m,n and r on N.) Let

$$(3.11) \qquad \Delta_N = \frac{1}{\sqrt{v_N}}$$

that is, $c_{Ni} = (1-z_i)/\ v_N$, $(i=1,2,\cdots,v_N)$ and assume that

$$(3.12) \qquad \lim_{N\to\infty} \frac{m_N}{v_N} = \lim_{N\to\infty} \frac{m}{N} = \lambda \quad .$$

From (3.11) and (3.12) it is easily verified that Assumption 3.2 is satisfied. We now define $\varphi_N(u)$ explicitly by asserting that it is a step function which is constant on the intervals $\frac{i-1}{v_N} < u \leq \frac{i}{v_N}$ and defined at the intermediate points $i/(v_N+1)$ for $i=1,2,\cdots,v_N$ by

$$(3.13) \qquad \varphi_N\left(\frac{i}{v_N+1}\right) = \begin{cases} \dfrac{i-r_N-1}{v_N} + \dfrac{(r_N+1)^2}{2v_N^{\ 2}} & \text{for } i \leq r_N \\ (r_{N+1})^2/2v_N^{\ 2} & \text{for } i > r_N \end{cases} \quad .$$

Hence from (3.11), (3.13) and (3.9) we obtain

$$(3.14) \qquad T_N = \sum_{i=1}^{r_N} \left(\frac{1-z_i}{v_N} - \frac{n_N}{v_N^{3/2}}\right)\frac{i-r_N-1}{v_N}$$

$$= \frac{m_N(r_N+1)}{2v_N^{\ 5/2}} - \frac{1}{\sqrt{v_N}}\, T_r^{(N)}$$

By corollary 2.2 of Hájek [17] , the statistic T_N is asymptotically normally distributed provided that assumption 3.1 is met. It follows from (3.14) that the same result holds for $T_r^{(N)}$. To show that Assumption 3.1 is satisfied we note that H is an arbitrary distribution function in $\mathcal{H}$, i.e., H satisfies (3.3). If we can first find a

$\varphi(u)$ satisfying (3.10) then, using remark 3.8 of Hájek, we can find an H in terms of φ. The function φ can be taken to be

$$(3.15)\qquad \varphi(u) = \begin{cases} u-p+p^2/2 , & 0<u\leq p \\ p^2/2 & u>p \end{cases}$$

where $p = \lim_{N\to\infty} \frac{r_N}{v_N} = \lim_{N\to\infty} \frac{r}{N}$.

Letting $X_0 = H^{-1}(p)$ denote the p^{th} fractile of H we then obtain

$$(3.16)\qquad h(x) = \begin{cases} \dfrac{2c^2\, e^{-c(x+k)}}{[1+e^{-c(x+k)}]^2} , & -\infty < x \leq X_0 \\ \dfrac{p^2}{2}\, e^{-x\, p^2/2} , & x > X_0 \end{cases} .$$

where $c = p-p^2/2$ and k is another constant depending only on p.

It should be noted that the asymptotic ($m,N\to\infty$ with $\frac{m}{N}\to\lambda$) normality of $T_r^{(N)}$ also follows from the Chernoff-Savage theorem [5] using the relaxed sufficiency conditions given in [16] . Here we again assume that the weight function $\varphi(u)$ is absolutely continuous and we also assume that there exists a $\lambda_0 \leq \frac{1}{2}$ such that $0<\lambda_0 \leq \lambda \leq 1-\lambda_0<1$, where λ is defined in (3.12). If $\lambda_N = m/N$ then $\lim_{N\to\infty} \lambda_N = \lambda$. Then by the Chernoff-Savage theorem if σ_N, defined below, is positive then

$$(3.17)\qquad \lim_{N\to\infty} P\left\{ \frac{(T_r^{(N)}/m)-\mu_N}{\sigma_N} < t \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2}\, du \quad ;$$

here μ_N and σ_N are given by

$$(3.18)\qquad \mu_N = \int_{-\infty}^{\infty} \varphi[H(x)]\, dF(x)$$

and

$$(3.19)\quad N\sigma_N^{2} = 2(1-\lambda_N)\{\iint_{-\infty<x<y<\infty} G(x)[1-G(y)]\,\varphi'[H(x)]\,\varphi'[H(y)]dF(x)dF(y)$$

$$+\frac{(1-\lambda_N)}{\lambda_N}\iint_{-\infty<x<y<\infty} F(x)[1-F(y)]\varphi'[H(x)]\,\varphi'[H(y)]dG(x)dG(y)\}$$

where $\varphi'[H(x)]$ denotes $\left.\frac{d\varphi(u)}{du}\right|_{\mu=H(x)}$ and

$$H(x) = \lambda F(x) + (1-\lambda)G(x) \quad .$$

If, for example, $G(x) = F(x-\theta_N)$ and $\theta_N \to 0$ as $N\to\infty$, then by Corollary 2 of Chernoff and Savage

$$(3.20)\quad \lim_{N\to\infty}\frac{N\lambda_N\sigma_N^2}{(1-\lambda_N)} = \int_0^1 \varphi^2(u)du - [\int_0^1 \varphi(u)du\,]^2 \quad .$$

For the particular statistic $T_r^{(N)}$, we obtain from (3.15) and (3.20)

$$(3.21)\quad \lim_{N\to\infty}\frac{N\lambda_N\sigma_N^2}{(1-\lambda_N)} = \frac{p^3}{12}(4-3p) \quad .$$

4. Consistency of the $T_r^{(N)}$ test

Consistency of the $T_r^{(N)}$ test of $H_0:F(x) = G(x)$ against one-sided alternatives $H_1:G(x) = F(x-\theta)$ for $\theta > 0$, say is shown by using the Chernoff-Savage theorem [5] . When the null hypothesis is not true, i.e., $\theta > 0$ it follows easily from (3.15), since $\lambda<1$, that

$$(4.1)\quad \varphi(F(x)) - \varphi(H(x)) = (1-\lambda)(F-G) \neq 0,$$

for all x in the interval $-\infty < x \leq \min[H^{-1}(p), F^{-1}(p)]$. It is also seen that

$$(4.2)\quad \int_{-\infty}^{\infty}[\varphi(F(x)) - \varphi(H(x))]dF(x) \neq 0$$

by writing the integral in (4.2) as the sum of three separate integrals formed by the intervals with endpoints $\pm\infty$, $F^{-1}(p)$ and $H^{-1}(p)$; in fact if $\theta > 0$ the result is negative and if $\theta < 0$ the result is positive.

Let $\sigma_N^2(\theta)$ denote the variance of $T_r^{(N)}$ under the alternative hypothesis $G(x) = F(x-\theta)$. As $N \to \infty$ and $\theta_N \to \theta_0 = 0$ it follows from (3.26) that $\sigma_N^{2}(\theta_0) \to 0$ and $\sigma_N^{2}(\theta_N) \to 0$. Using Chebyshev's inequality we find that the $T_r^{(N)}$ test, and consequently the equivalent $V_r^{(N)}$ test, are consistent against one-sided alternatives of the form $G(x) = F(x-\theta)$.

The consistency of the $T_r^{(N)}$ test for two-sided alternatives can be proved in a similar manner.

5. Efficacy of the $T_r^{(N)}$ test

For comparing the large sample power of two sequences of tests, the concept of asymptotic relative efficiency (ARE) was developed by Pitman [24] . An exposition of his work with extensions is given by Noether [23] , (see also Fraser [13]).

Let $\beta_N(\theta)$ and $\beta_N^*(\theta)$ denote the power functions of two tests, say T and T^*, respectively, based on the same set of N observations, against a parametric family of alternatives indexed by θ, and let θ_0 be the value of θ corresponding to the null hypothesis. We shall assume that all tests are at the same level of significance α. Let β be a specified power with $\alpha < \beta < 1$, i.e., the tests are unbiased. Consider a sequence of alternatives $\theta_N \to \theta_0$ at the rate $N^{-\frac{1}{2}}$ such that

(5.1) $\qquad \beta_N(\theta_N) \to \beta \qquad$ as $N \to \infty$,

and a sequence $N^* = u(N)$ which is a monotonic increasing function N such that

(5.2) $\qquad {\beta^*}_{N^*}(\theta_N) \to \beta \qquad$ as $N \to \infty$.

If

$$(5.3) \qquad e(T^*,T) = \lim_{N \to \infty} \frac{N}{N^*}$$

exists, and is independent of α, β and the particular sequences $\{\theta_N\}$ and $\{u(N)\}$ chosen to satisfy (5.1) and (5.2), then $e(T^*,T)$ is defined to be the asymptotic relative efficiency of the test T^* with respect to the test T.

Considering that under the alternative hypothesis F and G differ only by a shift of location (or scale) parameter, i.e., assuming that $G(x) = F(x-\theta)$ for $\theta \neq 0$ (or $G(x) = F(\theta x)$ for $\theta \neq 1$), Pitman shows that $e(T^*,T)$ can be evaluated both for the one-sided and two-sided case by the formula

$$(5.4) \qquad e(T^*,T) = \lim_{N \to \infty} \frac{\left[\left. \frac{dE(T_N^*)}{d\theta} \right|_{\theta=\theta_0} \right]^2 \Big/ \sigma_0^2(T_N^*)}{\left[\left. \frac{dE(T_N)}{d\theta} \right|_{\theta=\theta_0} \right]^2 \Big/ \sigma_0^2(T_N)} = \lim_{N \to \infty} \frac{e(T_N^*)}{e(T_N)} \quad ;$$

here $E(T_N)$ denotes the expectation of T_N under H_1 and $\sigma_0^2(T_N)$ denotes the variance of T_N under the null hypothesis and the efficacy $e(T_N)$ of a test T_N based on N observations is defined by

$$(5.5) \qquad e(T_N) = \left[\left. \frac{dE(T_N)}{d\theta} \right|_{\theta=\theta_0} \right]^2 \Big/ \sigma_0^2(T_N) \quad .$$

Thus any two tests can be compared if their efficacies (or limiting efficacies) are known. In this section we shall calculate the efficacy of the $T_r^{(N)}$ test.

For the statistic $T_r^{(N)}$ we have $ET_N = m\mu_N$ where μ_N is given by

(3.18); hence for the location parameter problem

$$(5.6) \qquad \lim_{N\to\infty} \frac{d\mu_N}{d\theta} = (1-\lambda)\int_{-\infty}^{\infty} \varphi'[H(x)]f(x-\theta)dF(x)$$

where $\varphi[H(x)]$ is given by (3.15). Using (3.20) and (5.6), the efficacy of $T_r^{(N)}$ is given by

$$(5.7) \qquad e(T_r^{(N)}) = \frac{12N\lambda(1-\lambda)}{p^3(4-3p)} \left\{ \int_{-\infty}^{F^{-1}(p)} f^2(x)dx \right\}^2 .$$

For p=1, the above reduces to

$$(5.8) \qquad 12N\lambda(1-\lambda)\left[\int_{-\infty}^{\infty} f^2(x)dx\right]^2 ,$$

which is the known value of the efficacy of the Wilcoxon statistic.

Similarly, for scalar alternatives if we assume $G(x) = F(\theta x)$, we have

$$(5.9) \qquad \left[\left.\frac{d\mu_N}{d\theta}\right|_{\theta=1} \right]^2 = (1-\lambda)^2 \left\{ \int_{-\infty}^{F^{-1}(p)} xf^2(x)dx \right\}^2 ,$$

so that the efficacy of the $T_r^{(N)}$ statistic is given by

$$(5.10) \qquad \frac{12N\lambda(1-\lambda)}{p^3(4-3p)} \left\{ \int_{-\infty}^{F^{-1}(p)} xf^2(x)dx \right\}^2 .$$

6. ARE of $T_r^{(N)}$ with respect to the F-test for the scale parameter in the cases of the exponential distribution

The $T_r^{(N)}$ test can also be used for testing the H_0:F=G against scalar alternatives H_0:G(x)=f(θx) where $\theta > 0$. This is possible for positive random variables since the scale parameter becomes the location parameter under logarithmic transformation and the rank tests remain invariant under any strictly increasing transformation of the original variables.

In this section we shall compute the ARE of the $T_r^{(N)}$ test with respect to the F-test (which is equivalent to the likelihood ratio test) when the underlying parent populations are exponential. The exponential distribution is the most widely used model in problems of life-testing. (See Epstein and Sobel [10] , [11]). Considering only positive random variables we let

$$F(x) = 1-e^{-x} \quad \text{for } x > 0 \tag{6.1}$$

and

$$G(y) = 1-e^{-\theta y} \quad \text{for } y > 0 \text{ where } \theta > 0. \tag{6.2}$$

The likelihood-ratio test of H_0: $\theta = \theta_0 = 1$ against alternative $\theta < 1$ is based on the first r ordered observations from a combined sample of m+n observations, (m are x's and n are y's), where $m_r \leq m$ observations are from (6.1) and $n_r \leq n$ observations are from (6.2), so that $m_r+n_r=r$. This test reduces to the F test conditioned on m_r and n_r [12] for which the test statistic is given by

$$R = \frac{m_r \{ \sum_{i=1}^{n_r} y_i + (n-n_r)y_{n_r} \}}{n_r \{ \sum_{i=1}^{m_r} x_i + (m-m_r)x_{m_r} \}} \tag{6.3}$$

follows the F-distribution with $(2n_r, 2m_r)$ degrees of freedom under H_0 and that $R\theta$ has the same distribution under the alternative hypothesis $G(x) = F(\theta x)$.

In the above case we have

$$F^{-1}(p) = -\log(1-p) . \tag{6.4}$$

Hence using (5.10) the efficacy of the $T_r^{(N)}$ statistic is given by

$$(6.5) \qquad e(T_r^{(N)}) = \frac{12N\lambda(1-\lambda)}{p^3(4-3p)} \left\{ \int_0^{F^{-1}(p)} x\, e^{-2x} dx \right\}^2$$

$$= \frac{3N\lambda(1-\lambda)}{4p^3(4-3p)} \{ 2(1-p)^2 \log(1-p) + 2p - p^2 \}^2 \quad .$$

It can be easily shown by using the above result that $R\theta$ has the F-distribution under the alternative hypothesis so that the efficacy of R is

$$(6.6) \qquad e(R) = \frac{n_r(m_r-2)}{r-1} \simeq \frac{n_r m_r}{r} \quad .$$

Hence from (5.4), (6.5) and (6.6) the ARE of the $T_r^{(N)}$ test with respect to the F-test, conditioned on m_r, n_r, is given by

$$(6.7) \qquad e(T_r^{(N)}, F_{2n_r, 2m_r}) = \lim_{N \to \infty} \frac{3(\frac{r}{N})\lambda(1-\lambda)}{4(\frac{m_r}{N})(\frac{n_r}{N})p^3(4-3p)} \{2(1-p)^2 \log(1-p) + 2p - p^2\}^2$$

$$= \frac{3p}{4(4-3p)} \{2(1-p)^2 \log(1-p) + 2p - p^2\}^2 \quad .$$

In deriving the last expression we have used the fact that under H_0 (or under the alternative hypothesis $\theta = \theta_N$ where $\theta_N \to \theta_0 = 1$) as $N \to \infty$ the ratio $\frac{m_r}{N}$ has the binomial distribution with expectation $\frac{rm}{N^2}$ which tends to λp and has variance $\frac{rmn}{N^4}$ which tends to zero; similarly $\frac{n_r}{N}$ tends to $(1-\lambda)p$ as $N \to \infty$.

In the special case p=1, we obtain from (6.7)

$$(6.8) \qquad e(T_r^{(N)}, F_{2n_r, 2m_r}) = .75 \quad .$$

7. Comparison with the Precedence Test

Recently Eilbott and Nadler [7] have studied the properties of a test called the precedence test which is a modification of the exceedance test proposed earlier by Epstein [8]. In this section we shall compute the ARE of the $T_r^{(N)}$ test with respect to the precedence test P which we now describe.

Let $F(x, \theta)$ denote a family of c.d.f's indexed by θ where θ is a pure location or a pure scale parameter; in the latter case θ is assumed to be positive. Let m units called x's following the life distribution $F(x, \theta_1)$ and n units called y's following the life distribution $G(y) = F(x, \theta_2)$ are put to test simultaneously for testing

$$(7.1) \qquad H_0: \ \theta_1 = \theta_2$$

against alternative H_1: $\theta_1 < \theta_2$. Let $x_{(m')}$ denote the m'-th ordered observation of x's and similarly $y_{(n')}$ denote the n'-th ordered observation among the y's. The precedence test is defined through the

Precedence rule: the test is terminated as soon as either m' of the x's or n' of the y's fail, whichever comes sooner. The null hypothesis is rejected if $x_{(m')} < y_{(n')}$; otherwise it is accepted.

Thus the maximum number of units observed is m'+n'-1. In order to make the test comparable with the $T_r^{(N)}$ test we take $m' = m_r$ and $n' = n_r$ (and drop the prime notation) so that m'+n'-1 = r-1. The test statistic for the precedence rule can be written as

$$(7.2) \qquad P = x_{(m_r)} - y_{(n_r)}$$

and the critical region is given by $P < 0$. To derive the efficacy of the test statistic P we first find the asymptotic distribution of P as $N = m+n \to \infty$ and $m/N \to \lambda$.

Define γ_m and δ_n by

$$(7.3) \qquad \begin{aligned} \frac{m_r}{m+1} &= \gamma_m + O\left(\frac{1}{m+1}\right) , \\ \frac{n_r}{n+1} &= \delta_n + O\left(\frac{1}{n+1}\right) ; \end{aligned}$$

we define the symbols x_{γ_m} and y_{δ_n} to be the γ-th and δ-th sample quantiles of the two distributions $F(x, \theta_1)$ and $F(x, \theta_2)$ respectively. We make the following assumption.

Assumption 7.1. Unique γ-th and δ-th population quantiles ξ_γ and η_δ exist for the distributions $F(x, \theta_1)$ and $F(x, \theta_2)$ respectively and $f(\xi_\gamma , \theta_1) > 0$, $f(\eta_\delta , \theta_2) > 0$.

Then, from a well-known result (see for example Cramér [6] , or Wilks [33]) we have

(7.4) and

$$X_{(m_r)} \sim N(\xi_\gamma , \frac{\gamma(1-\gamma)}{mf^2(\xi_\gamma , \theta_1)})$$

$$Y_{(n_r)} \sim N(\eta_\delta , \frac{\delta(1-\delta)}{nf^2(\eta_\delta , \theta_2)})$$

where $X \sim N(\mu , \sigma^2)$ denotes that X is normally distributed with mean μ and variance σ^2. Hence using the independence of X and Y

$$(7.5) \qquad P = X_{(m_r)} - Y_{(n_r)} \sim N(\xi_\gamma - \eta_\delta , \frac{\gamma(1-\gamma)}{mf^2(\xi_\gamma , \theta_1)} + \frac{\delta(1-\delta)}{nf^2(\eta_\delta , \theta_2)}) \quad .$$

Since the value of θ in $F(x, \theta_1)$ is fixed, the efficacy of the P-statistic is given by

$$(7.6) \qquad e(P) = \left[\frac{d\eta_\delta}{d\theta} \bigg|_{\theta = \theta_1} \right]^2 \Bigg/ \left[\frac{\gamma(1-\gamma)}{mf^2(\xi_\gamma , \theta_1)} + \frac{\delta(1-\delta)}{nf^2(\eta_\delta , \theta_1)} \right] \quad .$$

The special case $\gamma = \delta$ is of particular interest. Letting

$\sigma^2 = \dfrac{\delta(1-\delta)}{f^2(\eta_\delta , \theta_1)}$ we have

$$(7.7)\qquad \sigma_0^2(P) = \frac{\delta(1-\delta)}{f^2(\eta_\delta\ ,\ \theta_1)}\left(\frac{1}{m}+\frac{1}{n}\right)$$

$$= \sigma^2\frac{N}{mn} \simeq \frac{\sigma^2}{N\lambda\ (1-\lambda\ )}\quad .$$

Hence the ARE of $T_r^{(N)}$ with respect to P for comparing location is

$$(7.8)\qquad e(T_r^{(N)},P) = \lim_{N\to\infty} \frac{12N\lambda(1-\lambda)}{p^3(4-3p)}\left\{\int_{-\infty}^{F^{-1}(p)} f^2(x)dx\right\}^2 \Bigg/ \frac{\left\{\frac{d\eta_\delta}{d\theta}\Big|_{\theta=\theta_1}\right\}^2}{\sigma^2/\{N\lambda\ (1-\lambda)\}}$$

$$= \frac{12\ \sigma^2}{p^3(4-3p)} \cdot \frac{\left\{\int_{-\infty}^{F^{-1}(p)} f^2(x)dx\right\}^2}{\left\{\frac{d\eta_\delta}{d\theta}\Big|_{\theta=\theta_1}\right\}^2}\quad .$$

In particular if θ is the population median then we will take $\gamma = \delta = \frac{1}{2}$ so that $\eta_\delta = \eta_{\frac{1}{2}} = \theta$ and we have

$$(7.9)\qquad e(T_r^{(N)},P) = \frac{12\ \sigma^2}{p^3(4-3p)}\left\{\int_{-\infty}^{F^{-1}(p)} f^2(x)dx\right\}^2\quad .$$

For p=1 equation (7.9) reduces to

$$(7.10)\qquad 12\,\sigma^2\left[\int_{-\infty}^{\infty} f^2(x)dx\right]^2\ ;$$

this is essentially the same expression as that derived by Pitman [24] for comparing the Wilcoxon test with the t-test. In Table I (7.8) has been evaluated for the normal, rectangular, exponential, gamma and the Weibull distribution. In the last two cases to evaluate (7.8) numerically we need to know the value of the shape parameter t. It can be easily checked that for all the interesting cases $e(T_r^{(N)},P) > 1$. Thus the $T_r^{(N)}$ test performs better than the precedence test in a large number of interesting cases. Since $T_r^{(N)}$ is equivalent to $V_r^{(N)}$ the same holds for the $V_r^{(N)}$ test.

It is interesting to see how the $T_r^{(N)}$ test compares with the precedence test when applied to scalar alternatives, where θ denotes the parameter of the exponential distribution. If $\theta_0=1$ and

$$G(x, \theta_1) = 1-e^{-\theta_1 x} \quad \text{for } x>0$$

then

$$(7.11) \qquad \left.\frac{d\eta_\delta}{d\theta_1}\right|_{\theta_2=\theta_1=1} = \log(1-\delta) \ .$$

Hence from (6.5), (7.6) and (7.11) we have

$$(7.12) \qquad e(T_r^{(N)}P) = \frac{3\sigma^2}{4p^3(4-3p)} \frac{\{2(1-p)^2\log(1-p)+2p-p^2\}^2}{\{\log(1-\delta)\}^2}$$

Furthermore, if we take $\delta = (1-e^{-1})$ so that $\eta_\delta = \frac{1}{\theta}$, the mean of the distribution. In this case (7.12) becomes

$$(7.13) \qquad e(T_r^{(N)},P) = \frac{3(e-1)}{4p^3(4-3p)} \{2(1-p)^2\log(1-p)+2p-p^2\}^2 \quad ,$$

and this quantity has been evaluated in Table II for different values of p. For scalar alternatives it can be seen from this table that the $T_r^{(N)}$ test is superior to the precedence test for $p> .75$ and performs comparably with it for as low as $p = .5$.

8. Comparison with other a.m.p.r. tests from censored data.

In a recent paper Gastwirth [14] has considered several rank tests based on well-known statistics for the case of censored data and he has derived the weight functions J(u) for which these tests are asymptotically most powerful rank tests (a.m.p.r.t.). His weight functions for the modified Wilcoxon test is

$$(8.1)\qquad J(u) = \begin{cases} u - \frac{1}{2} & 0 \leq u \leq p \\ p/2 & p < u \leq 1 \end{cases}$$

where p has the same meaning as before; we denote the corresponding statistic for this test by $G_r^{(N)}$. From the discussion of the asymptotic normality of $T_r^{(N)}$ and from theorem 1.1 of Hájek [17] it is clear that each of the statistics based on weight functions $\varphi(u)$ in (3.15) and J(u) in (8.1) is an a.m.p.r.t. with respect to a certain family of distributions. In each case the family has the form of a logistic distribution to the left of the censored percentile and has the exponential form to the right of this point. However, the two families differ in functional form, the first one being given by (3.16) and the second is given by letting $X_0 = H^{-1}(p)$,

$$(8.2)\qquad h(x) = \begin{cases} \dfrac{\frac{1}{2}\, e^{-\frac{1}{2}(x+k)}}{(1+e^{-\frac{1}{2}(x+k)})^2}, & -\infty < x \leq X_0 \\ \frac{p}{2}\, e^{-px/2} & x > X_0 \end{cases}$$

where k is a function of p.

It is of interest to find the ARE of one test when the other is the a.m.p.r.t. We can take this in either direction since the result is symmetric. From section 6 of Hájek [17] it is known that the ARE of statistic T_1 with respect to T_2 is given by

$$(8.3)\qquad e(T_1,T_2) = \rho^2(T_1,T_2)$$

where T_2 corresponds to the a.m.p.r. test for the underlying distribution, T_1 corresponds to any other rank test and $\rho(T_1,T_2)$ is the correlation coefficient between T_1 and T_2 and in our case is given by

$$(8.4) \qquad e(T_r^{(N)},G_r^{(N)}) = \rho^2(T_r^{(N)},G_r^{(N)}|p)$$

$$= \frac{[\int_0^1 \varphi(u)J(u)du]^2}{\int_0^1 \varphi^2(u)du \int_0^1 J^2(u)du}$$

$$= \frac{(3-2p)^2}{(4-3p)(3-3p+p^2)} .$$

It can be easily seen that $\rho^2(T_r^{(N)},G_r^{(n)}|p)$ is an increasing function of p so that $\rho^2(T_r^{(N)},G_r^{(N)}|p) \geq \rho^2(T_r^{(N)},G_r^{(N)}|0) = 0.75$. The ARE $e(T_r^{(N)},G_r^{(N)})$ has been computed for several values of p in table II; in particular for $p = \frac{1}{2}$, $\rho^2(T^{(N)},G^{(N)}|p) = .91$. We see from table II that for all p the performance of $G_r^{(N)}$ and $T_r^{(N)}$ are roughly comparable. However, $T_r^{(N)}$ has some additional advantages which $G_r^{(N)}$ does not have. One advantage of $T_r^{(N)}$ is that it can easily be put in the curtailed form as explained in [29] .

A second statistic worth comparing with the $T_r^{(N)}$ statistic is a form of the statistic $S_N^{(N)}$ proposed by Savage [27] which is the a.m.p.r.t. in the exponential as well as in the weibull case. For a fair comparison we shall consider the modified test $S_r^{(N)}$ based on censored data as given by Gastwirth [14] with weight function

$$(8.5) \qquad k(u) = \begin{cases} -\ln(1-u)-1) & 0 \leq u \leq p \\ -\ln(1-p) & p < u \leq 1 \end{cases} .$$

The ARE of $T_r^{(N)}$ with respect to $S_r^{(N)}$ when the underlying population is exponential from zero to the point of censoring and again exponential to the right of the point of censoring is given by

$$(8.6)\qquad e(T_r^{(N)},S_r^{(N)}) = \rho^2(T_r^{(N)},S_r^{(N)}) = \frac{\{\int_0^1 \varphi(u)k(u)du\ \}^2}{\int_0^1 \varphi^2(u)du \int_0^1 k^2(u)du}\ .$$

Now, using (3.21) and (8.5)

$$\int_0^1 \varphi^2(u)du = \frac{p^3}{12}(4-3p),\quad \int_0^1 k^2(u)du = p$$

and

$$\int_0^1 \varphi(u)k(u)du = \frac{1}{4}\ [2(1-p)^2\ln(1-p)+2p-p^2]\ .$$

Hence

$$e(T_r^{(N)},S_r^{(N)}) = \frac{3[2(1-p)^2\ln(1-p)+2p-p^2\ ]^2}{4p^4(4-3p)}$$

which is exactly the same expression (6.7) we get for the ARE of $T_r^{(N)}$ with respect to the likelihood ratio test. If $p \to 1$ the above ARE $\to .75$ implying a correlation coefficient of $\sqrt{.75} = .8660$ which agrees with Savage's result [27] .

In table II we have computed the ARE's of $T_r^{(N)}$ statistic with respect to the $G_r^{(N)}$ and $S_r^{(N)}$ or R statistics for different values of p when the latter statistics are optimal.

Rao, Savage and Sobel [26] have proposed another statistic $R_r^{(N)}$ for the case of censored data which is locally most powerful for the Lehmann family of alternatives (i.e., alternatives of the form $1-G = (1-F)^\theta$) and is the same as $T_N^{(N)}$ when the complete sample is available. We shall not compare $T_r^{(N)}$ with $R_r^{(N)}$ since asymptotic properties of $R_r^{(N)}$ are not known.

9. Conclusion

From the above discussions and from tables I and II we see that for testing for location alternatives the $V_r^{(N)}$ test is quite satisfactory for most of the cases encountered. Even for testing for scalar alternatives (especially for distributions useful in life testing) the performance of the $V_r^{(N)}$ test is reasonably satisfactory; however Savage's statistic appears to be the most suitable one for the case of scalar alternatives. Properties of the Savage statistic are being studied further and the results will be communicated later. Several k-sample extensions of the $V_r^{(N)}$ statistic will be considered in the next part of the paper.

Part II

The k-sample case

Chapter III

ON A GENERALIZED KRUSKAL STATISTIC

1. Introduction

Let X_{ij} $(j=1,2,\cdots,n_i;\ i=1,2,\cdots,k)$ be k independent samples of sizes $n_1,n_2,\cdots,n_k$ respectively from k populations with continuous cumulative distribution functions $F_1,F_2,\cdots,F_k$ respectively. We assume that the F_i's belong to a family of distribution functions $\mathcal{F}$ indexed by a parameter θ. Let all the $N = \sum_{i=1}^{k} n_i$ observations be put together and ordered to form a single sequence and suppose that only the first r ordered observations are available. That is, let us have a combined (right) censored sample of total size r. To test the hypothesis

$$H_o:\ F_1(x) = F_2(x) = \cdots = F_k(x) \tag{1.1}$$

(or equivalently, H_o: $\theta_1 = \theta_2 = \cdots = \theta_k = 0$ say, under location alternatives)

against the alternative hypothesis

$$H_1: = F_i(x) = F(x,\ \theta_i) \qquad (i=1,2,\cdots,k)$$

we propose a statistic $B_r^{(N)}$ (large values being critical) based on only the first r ordered observations from the combined sample. In section 2 we define the statistic $B_r^{(N)}$ and show its relationship with other statistics. The mean and variance of $B_r^{(N)}$ under the null hypothesis is derived in section 3. In section 4 we find the asymptotic distribution of $B_r^{(N)}$ both under the null and the non-null case. The computation of $B_r^{(N)}$ has been illustrated by an example in section 5.

2. Relationship of $B_r^{(N)}$ with Other Statistics

The statistic $B_r^{(N)}$ is defined below. Let the combined N observations be ordered and define

$$(2.1) \qquad z_\alpha^{(i)} = \begin{cases} 1 & \text{if the } \alpha^{th} \text{ ordered observation is from the } i^{th} \text{ population} \\ 0 & \text{otherwise } (\alpha=1,2,\cdots,N;\ i=1,2,\cdots,k). \end{cases}$$

The statistic $B_r^{(N)}$ is defined by

$$(2.2) \qquad B_r^{(N)} = G \sum_{i=1}^{k} \frac{1}{n_i} \left(S_i + \frac{r(r+1)}{2N^2} n_i\right)^2$$

which depends only on the first r ordered observations from the combined sample; here

$$(2.3) \qquad S_i = \sum_{\alpha=i}^{r} \left(\frac{\alpha-r-1}{N}\right) z_\alpha^{(i)} \qquad (i=1,2,\cdots,k)$$

and

$$(2.4) \qquad G = \frac{12\, N^3(N-1)}{r(r+1)[2N(2r+1)-3r(r+1)]} \quad .$$

We define n_{ir} as the cumulative number of failures from the i^{th} population among the first r failures and R_{ir} as the sum of the ranks of these n_{ir} failures. Clearly then we have both

$$(2.5) \qquad \sum_{i=1}^{k} n_{ir} = r, \qquad \sum_{i=1}^{k} R_{ir} = \frac{r(r+1)}{2}$$

and $\quad n_i = n_{iN} \quad (i=1,2,\cdots,k).$

In terms of the quantities

$$(2.6) \qquad n_{ir} = \sum_{\alpha=1}^{r} z_\alpha^{(i)}, \qquad R_{ir} = \sum_{\alpha=1}^{r} \alpha z_\alpha^{(i)}$$

we can rewrite (2.3) as

$$NS_i = R_{ir} - (r+1)n_{ir} \quad . \tag{2.7}$$

Substituting the value of S_i from (2.7) in (2.2) we obtain

$$B_r^{(N)} = G \sum_{i=1}^{k} \left(\frac{R_{ir}}{n_i} - (r+1)\frac{n_{ir}}{n_i} + \frac{r(r+1)}{2N}\right)^2 \quad . \tag{2.8}$$

Putting r=N in (2.7) we get

$$B_N^{(N)} = \frac{12}{N(N+1)} \sum_{i=1}^{k} n_i(\bar{R}_i - \frac{N+1}{2})^2 \quad , \tag{2.9}$$

which is the H-statistic proposed by Kruskal [ ] and is related to Terpstra's k-sample statistic [ ] ; here $\bar{R}_i = R_i/n_i$ and $R_i = R_{iN}$ is the sum of ranks of all the n_i observations from the i^{th} population. If we sum both sides of (2.7) we easily obtain

$$\sum_{i=1}^{k} S_i = - \frac{r(r+1)}{2N} \quad . \tag{2.10}$$

Using (2.10), equation (2.2) can be written in the form

$$B_r^{(N)} = G \left[\sum_{i=1}^{k} \frac{S_i^2}{n_i} - \frac{r^2(r+1)^2}{4N^3} \right] \quad . \tag{2.11}$$

For the special case k=2 we obtain from (2.11)

$$B_r^{(N)} = \frac{NG}{n_1 n_2} \left(S_1 + \frac{r(r+1)n_1}{2N^2}\right)^2 \quad . \tag{2.12}$$

The statistic proposed by Sobel [ ] is defined by

$$V_r^{(N)} = \sum_{i=1}^{r} (n_2 n_{1i} - n_1 n_{2i}) \tag{2.13}$$

and it has been shown in chapter II above that

$$T_r^{(N)} = S_1 - \frac{(r+1)^2 n_i}{2N^2} = \frac{m(r+1)}{2N^2} - \frac{V_r^{(N)}}{N^2} \quad . \tag{2.14}$$

Hence by (2.12) and (2.14) the two-tailed tests based on any one of the four statistics $B_r^{(N)}$, S_1, $T_r^{(N)}$ and $V_r^{(N)}$ are all equivalent for k=2.

3. Mean and Variance of $B_r^{(N)}$ under the Null Hypothesis

In this section we shall find the mean and variance of $B_r^{(N)}$ under the null hypothesis.

Let $\{X_{ij}\}$ be a random permutation of N fixed numbers $\{b_{ij}, j=1,2,\ldots,n_i;\ i=1,2,\cdots,k;\ \sum_{i=1}^{k} n_i = N\}$ such that each permutation is equally likely to occur. We also define k additional associated sequences $\{a_\alpha^{(i)}\}$ $(i=1,2,\cdots,k)$ with N fixed numbers in each and to each of the N! possible observations (i.e., permutations) on $\{X_{ij}\}$ we assign in a 1:1 manner a fixed permutation in each of the associated $\{a_\alpha^{(i)}\}$ sequences $(i=1,2,\cdots,k)$. Let the random sequences associated with $\{a_\alpha^{(i)}\}$ be denoted by $\{\xi_\alpha^{(i)}\}$, so that the one sequence of random variables $\{X_{ij}\}$ is associated with k sequences of random variables $\{\xi_\alpha^{(i)}\}$ $(i=1,2,\cdots,k)$. For each i we define k linear statistics

$$Y_i = \sum_{\alpha=1}^{N} c_\alpha \xi_\alpha^{(i)} \qquad (i=1,2,\cdots,k) \tag{3.1}$$

where the coefficients c_α's are arbitrary constants. Let $\bar{c} = \frac{1}{N}\sum_{\alpha=1}^{N} c_\alpha$, $\bar{a}^{(i)} = \frac{1}{N}\sum_{\alpha=1}^{N} a_\alpha^{(i)}$ $(i=1,2,\cdots,k)$ and let $E(Y_i)$, $\sigma^2(Y_i)$ and $\sigma(Y_i,Y_j)$ denote as usual the mean of Y_i, the variance of Y_i and the covariance between Y_i and Y_j respectively $(i,j=1,2,\cdots,k)$. We now prove the following useful theorem.

<u>Theorem 3.1</u>: For the above structure with Y_i defined in (3.1)

$$E(Y_i) = N\bar{c}\,\bar{a}^{(i)} \quad , \tag{3.2}$$

$$\sigma^2(Y_i) = \frac{1}{N-1} \sum_{\alpha=1}^{N} (c_\alpha - \bar{c})^2 \sum_{\beta=1}^{N} (a_\beta^{(i)} - \bar{a}^{(i)})^2 \quad , \tag{3.3}$$

and

$$\sigma(Y_i, Y_j) = \frac{1}{N-1} \sum_{\alpha=1}^{N} (c_\alpha - \bar{c})^2 \sum_{\beta=1}^{N} (a_\beta^{(i)} - \bar{a}^{(i)})(a_\beta^{(j)} - \bar{a}^{(j)}) \tag{3.4}$$

$$(i, j = 1, 2, \cdots, k).$$

<u>Proof</u>. Result (3.2) is obvious. To prove (3.4) we have for any pair (i,j)

$$\sigma(Y_i, Y_j) = \sum_{\alpha=1}^{N} c_\alpha^2 \,\sigma(\xi_\alpha^{(i)}, \xi_\alpha^{(j)}) + \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{N} c_\alpha c_\beta \quad \sigma(\xi_\alpha^{(i)}, \xi_\beta^{(j)})$$

$$= \sum_{\alpha=1}^{N} c_\alpha^2 \left\{ \sum_{\beta=1}^{N} \frac{a_\beta^{(i)} a_\beta^{(j)}}{N} - \bar{a}^{(i)}\bar{a}^{(j)} \right\}$$

$$+ \left\{ (\sum_{\alpha=1}^{N} c_\alpha)^2 - \sum_{\alpha=1}^{N} c_\alpha^2 \right\} \left\{ \frac{\sum_{\beta\neq\gamma=1}^{N} a_\beta^{(i)} a_\gamma^{(j)}}{N(N-1)} - \bar{a}^{(i)}\bar{a}^{(j)} \right\}$$

$$= \sum_{\alpha=1}^{N} c_\alpha^2 \left\{ \sum_{\beta=1}^{N} \frac{a_\beta^{(i)} a_\beta^{(j)}}{N} - \bar{a}^{(i)}\bar{a}^{(j)} \right\}$$

$$+ \left\{ (\sum_{\alpha=1}^{N} c_\alpha)^2 - \sum_{\alpha=1}^{N} c_\alpha^2 \right\} \left\{ \frac{\bar{a}^{(i)}\bar{a}^{(j)}}{(N-1)} - \frac{\sum_{\beta=1}^{N} a_\beta^{(i)} a_\beta^{(j)}}{N(N-1)} \right\}$$

$$= \frac{1}{N-1} \sum_{\alpha=1}^{N} (c_\alpha - \bar{c})^2 \sum_{\beta=1}^{N} (a_\beta^{(i)} - \bar{a}^{(i)})(a_\beta^{(j)} - \bar{a}^{(j)}) \quad .$$

Proof of (3.3) follows from that of (3.4) by taking i=j.

In tying the above theorem up with our problem it may be noted that the null hypothesis corresponds to the case in which all permutations of $\{X_{ij}\}$ are equally likely. Hence letting $E_o(\cdot)$, $\sigma_o^2(\cdot)$, $\sigma_o(\cdot,\cdot)$ denote the mean variance and covariance under H_o we have the following corollaries.

Corollary 3.1. Under the null hypothesis H_o

$$(3.5)\qquad E_o(S_i) = -\frac{r(r+1)}{2N^2}\, n_i \quad ,$$

$$(3.6)\qquad \sigma_o^2(S_i) = \frac{n_i(N-n_i)}{GN} \quad ,$$

and

$$(3.7)\qquad \sigma_o(S_i,S_j) = \frac{-n_i n_j}{GN} \qquad (i,j=1,2,\cdots,k;\ i\neq j) \ .$$

Proof. The results directly follow from theorem 3.1 by taking

$$(3.8)\qquad c_\alpha' = \begin{cases} \dfrac{\alpha-r-1}{N} \quad , & 1 \leq \alpha \leq r \\ 0 \quad , & r+1 \leq \alpha \leq N \end{cases}$$

and the definition of $z_\alpha^{(i)}$ as given in (2.1).

Corollary 3.2. Under the null hypothesis

$$(3.9)\qquad E_o(B_r^{(N)}) = k-1 \quad .$$

Proof.

$$E_o(B_r^{(N)}) = G \sum_{i=1}^{k} \frac{1}{n_i} E(S_i - ES_i)^2$$

$$= k-1.$$

It is important to notice that the expected value of $B_r^{(N)}$ is independent of r, the point of censoring.

To find the variance of $B_r^{(N)}$ under the null hypothesis we first compute $E(B_r^{(N)})^2$. The details of the computation is given in appendix I. From appendix I we find that

$$(3.10) \qquad \frac{N^4 E_o(B_r^{(N)})^2}{G^2} = \frac{(r+1)^{(2)}(6r^3+9r^2+r-1)}{30N} \sum_{i=1}^{k} \frac{1}{n_i}$$

$$+ \frac{(r+1)^{(5)}(15r^3+15r^2-10r-8)}{240N^{(4)}}(N^2-2Nk-4N+k^2+10k-6 \sum_{i=1}^{k} \frac{1}{n_i})$$

$$+ \frac{(r+1)^{(4)}(30r^3+35r^2-11r-12)}{180N^{(3)}}(Nk+2N-k^2-8k+6 \sum_{i=1}^{k} \frac{1}{n_i})$$

$$+ \frac{(r+1)^{(3)}}{180N^{(2)}} [(20r^3+24r^2-5r-6)k^2+(130r^3+174r^2-10r-36)k - (150r^3+198r^2-15r-42) \sum_{i=1}^{k} \frac{1}{n_i}]$$

$$- \frac{r^3(r+1)^3}{48N^2(N-1)} [3Nr^2+3r^2-5Nr+3r-4N+(8Nr-6r^2+4N-6r)k] .$$

From (3.10) we can compute $E_o(B_r^{(N)})^2$ and

$$(3.11) \qquad \sigma_o^2(B_r^{(N)}) = E_o(B_r^{(N)})^2 - (k-1)^2 .$$

As a check on the computations we put r=N in (3.11) and deduce, after some simplification,

$$(3.12) \qquad \sigma_o^2(B_N^{(N)}) = 2(k-1) - \frac{2}{5N(N+1)} [3k^2-6k+N(2k^2-6k+1)] - \frac{6}{5} \sum_{i=1}^{k} \frac{1}{n_i} ,$$

which is the result obtained by Kruskal [21] .

4. Asymptotic Distribution of $B_r^{(N)}$

In this section we shall find the asymptotic distribution of $B_r^{(N)}$ when N, n_i and r all become infinitely large in such a way that

$$(4.1) \qquad \lim_{N\to\infty} \frac{r}{N} = p > 0, \quad \lim_{N\to\infty} \frac{n_i}{N} = \lambda_i \qquad (i=1,2,\cdots,k)$$

where $0 < \lambda_o \leq \lambda_i \leq 1- \lambda_o < 1$ $(i=1,2,\cdots,k)$ and λ_o is a constant not greater than $1/k$. Asymptotic normality of $B_r^{(N)}$ will be shown with the help of the k-sample version of the Chernoff-Savage theorem [5] as given by Puri [25] with the relaxed sufficiency conditions given by Govindarajulu, LeCam and Raghavachari [16] . We accomplish this by showing that $B_r^{(N)}$ is an L-statistic as defined in [25] .

We shall give Puri's theorem under the conditions given in [16] and show how the theorem applies to our case. Define

$$H(x) = \sum_{i=1}^{k} \lambda_i F_i(x) \tag{4.2}$$

and

$$H_N(x) = \sum_{i=1}^{k} \lambda_i F_i(x;n_i) \tag{4.3}$$

where $F_i(x;n_i)$ is the empirical cdf based on $x_{i1}, x_{i2}, \cdots, x_{in_i}$ and H_N then denotes a combined sample cdf with weights λ_i. Now define

$$T_{N,i} = n_i^{-1} \sum_{\alpha=1}^{N} E_{N,\alpha}^{(i)} z_{N,\alpha}^{(i)} \tag{4.4}$$

where the $E_{N,\alpha}^{(i)}$'s are given numbers, and $z_{N,\alpha}^{(i)}$ have been defined before as $z_{N,\alpha}^{(i)} \equiv z_\alpha^{(i)}$. (The additional subscript N in $z_{N,\alpha}^{(i)}$ is needed to study the asymptotic properties as $N \to \infty$.) We can represent $T_{N,i}$ by

$$T_{N,i} = \int_{-\infty}^{\infty} J_{N,i}\left(\frac{NH_N}{N+1}\right) dF_i(x;n_i) \qquad (i=1,2,\cdots,k) \tag{4.5}$$

where $J_{N,i}(u)$ is an arbitrary weight function defined on the open interval (0,1). We shall use $J_i^{(j)}(H)$ for j=0,1, to denote, respectively $J_i(H)$ and the first derivative $J_i'(H)$ of $J_i(H)$. Puri's theorem with relaxed conditions as given in [16] can now be stated.

Theorem 4.1. If for each i (i=1,2,$\cdots$,k) we have the four conditions:

(a) $$J_i(H) = \lim_{N\to\infty} J_{N,i}(H)$$

exists for $0<H<1$, is not a constant, and is absolutely continuous,

(b) $$\int_{0<H_N\leq 1} [J_{N,i}(\frac{NH_N}{N+1}) - J_i(\frac{NH_N}{N+1})dF_i(x;n_i) = 0_p(N^{\frac{1}{2}}) ,$$

(c) $$| J_i^{(j)}(H) | \leq M [H(1-H)]^{-j-\frac{1}{2}+\delta}$$

for $j=0,1$ and some $\delta > 0$; here M is a generic constant,

(d) the quantity $\sigma_{N,j}^2$ defined in (4.8) below is positive,

then the random vector

(4.6) $$\sqrt{N}(T_{N,1} - \mu_{N,k}),\cdots, \sqrt{N}(T_{N,k} - \mu_{N,k})$$

where

(4.7) $$\mu_{N,i} = \int_{-\infty}^{\infty} J_i(H(x))dF_i(x)$$

has a k-variate limiting normal distribution with mean vector zero and covariance matrix given by

$$\Sigma = (\sigma_{N,i,j})_{i,j=1,2,\cdots,k}$$

where

(4.8) $$\begin{aligned}\sigma_{N,i,i} &= N\sigma^2_{N,i}\\ &= \sum_{\substack{j=1\\ j\neq i}}^{k} 2\lambda_j \int_{-\infty<x<y<\infty} F_j(x)[1-F_j(y)]J_i'(H(x))J_i'(H(y))dF_i(x)dF_i(y)\\ &\quad + \frac{2}{\lambda_i}\int_{-\infty<x<y<\infty} F_i(x)[1-F_i(y)]J_i'(H(x))J_i'(H(y))d[H(x)-\lambda_iF_i(x)]\\ &\qquad d[H(y)-\lambda_iF_i(y)]\end{aligned}$$

and

$$(4.9) \qquad \sigma_{N,i,j} = E\{N(T_{N,i}-\mu_{N,i})(T_{N,j}-\mu_{N,j})\}$$

$$= \sum_{\substack{t=1\\ t\neq i,j}}^{k} \Big[\int\int_{-\infty<x<y<\infty} F_t(x)\{1-F_t(y)\} J_i'(H(x))J_j'(H(y))dF_j(x)dF_j(y)$$

$$+ \int\int_{-\infty<x<y<\infty} F_t(x)\{1-F_t(y)\} J_i'(H(y))J_j'(H(x))dF_i(y)dF_j(x) \Big]$$

$$- \int\int_{-\infty<x<y<\infty} F_i(x)\{1-F_i(y)\} J_i'(H(y))J_j'(H(x))dF_j(x)d\{H(y)-\lambda_i F_i(y)\}$$

$$- \int\int_{-\infty<x<y<\infty} F_i(x)\{1-F_i(y)\} J_i'(H(x))J_j'(H(y))dF_j(y)d\{H(x)-\lambda_i F_i(x)\}$$

$$- \int\int_{-\infty<x<y<\infty} F_j(x)\{1-F_j(y)\} J_i'(H(x))J_j'(H(y))dF_i(x)d\{H(y)-\lambda_j F_j(y)\}$$

$$- \int\int_{-\infty<x<y<\infty} F_j(x)\{1-F_j(y)\} J_i'(H(y))J_j'(H(x))dF_i(y)d\{H(x)-\lambda_j F_j(x)\} .$$

In our application the weight functions $E_{N,\alpha}^{(i)}$ will not depend on i (so that $J_i = J$, say, for each i) and in this case the random vector given in (4.6) follows a (k-1) variate nonsingular normal distribution. We now define

$$(4.10) \qquad A^2 = \int_0^1 J^2(u)du - \left(\int_0^1 J(u)du\right)^2$$

$$(4.11) \qquad W_i = n_i^{1/2}(T_{N,i} - \mu_{N,i}(\theta))/A \qquad (i=1,2,\cdots,k)$$

and denoting by $\mu_{N,i}(\theta)$ the mean of $T_{N,i}$ when $F_i(x) = F(x;\theta_i)$

$$(4.12) \qquad L = \sum_{i=1}^{k} W_i^2 = \sum_{i=1}^{k} \{ n_i^{\frac{1}{2}}(T_{N,i}-\mu_{N,i}(\theta))/A \}^2 .$$

It follows from the normality result above that the limiting distribution of L is a chi-square (χ^2_{k-1}) with k-1 degrees of freedom.

To show the asymptotic normality of $B_r^{(N)}$ all we need to show is that the conditions (a),(b),(c) and (d) of theorem 4.1 are met and that $B_r^{(N)}$ is an L-statistic as defined in (4.12). It can easily be seen that condition (d) holds for $N=\infty$ from (4.11) and (4.15) below. We define the statistic $T_{N,i}$ by

$$(4.13) \qquad n_i T_{N,i} = \sum_{\alpha=1}^{r} \left(\frac{\alpha-r-1}{N} + \frac{(r+1)^2}{2N^2}\right) z_\alpha^{(i)} + \sum_{\alpha=r+1}^{N} \frac{(r+1)^2}{2N^2} z_\alpha^{(i)} ,$$

$$= \sum_{\alpha=1}^{r} \left(\frac{\alpha-r-1}{N}\right) z_\alpha^{(i)} + \frac{(r+1)^2 n_i}{2N^2} ,$$

or equivalently

$$T_{N,i} = \int_{-\infty}^{\infty} J_N[H_N(x)] \, dF_i(x;n_i) \qquad (i=1,2,\cdots,k) .$$

Clearly then by (3.5)

$$(4.14) \qquad {}_{N,i}(0) = E(S_i) + \frac{n_i(r+1)^2}{2N^2} = \frac{r+1}{2N^2} .$$

From (4.13) we easily see that

$$(4.15) \qquad \lim_{N\to\infty} J_N(u) = J(u) = \begin{cases} u-p+p^2/2 & 0 \leq u \leq p \\ p^2/2 & u \geq p \end{cases}$$

which can be seen to satisfy conditions (a),(b) and (c) of theorem 4.1. Also we note from (4.15) that

$$(4.16) \qquad \int_0^1 J(u)du = 0 .$$

Now

$$(4.17)\qquad B_r^{(N)} = G \sum_{i=1}^{k} n_i\left(\frac{S_i}{n_i} + \frac{r(r+1)}{2N^2}\right)^2$$

$$= \sum_{i=1}^{k} \{n_i^{\frac{1}{2}} (T_{N,i} - \mu_{N,i}(0))/A_N\}^2 ,$$

which is of the form of an L-statistic where $A_N^2 = G^{-1}$ and

$$\lim_{N\to\infty} A_N = A = \int_0^1 J^2(u)du = \frac{p^3(4-3p)}{12} .$$

Hence, under the H_o the statistic $B_r^{(N)}$ is asymptotically distributed as a χ^2_{k-1} with (k-1) degrees of freedom. It follows directly from Puri [25] that under the alternative hypothesis H_1: $F_i(x) = F(x, \theta_i)$ where the θ_i's are not all equal the limiting distribution of $B_r^{(N)}$ will be a noncentral χ^2_{k-1} with (k-1) degrees of freedom with non-centrality parameter given by

$$\lambda(H;L) = \frac{12}{p^3(4-3p)} \sum_{i=1}^{k} \lim_{N\to\infty}\left[n_i^{\frac{1}{2}} \int_{-\infty}^{F^{-1}(p)} \sum_{\alpha=1}^{k} \lambda_\alpha \{F(x+ \frac{\theta_\alpha - \theta_i}{N}) - F(x)\} dF(x)\right]^2$$

In the special case where p=1, the above reduces to

$$\frac{12}{N^2} \sum_{i=1}^{k} n_i \sum_{\alpha=1}^{k} n_\alpha \left[\lim_{N\to\infty} \int_{-\infty}^{\infty} N\{F(x + \frac{\theta_\alpha - \theta_i}{N^{\frac{1}{2}}}) - F(x)\} dF(x)\right]^2$$

which is the corresponding expression for the noncentrality parameter associated with the Kruskal statistic H and was first derived by Andrews [1] .

5. An Example

The following example will illustrate the computation of $B_r^{(N)}$. In a bio-assay problem a certain drug is being administered simultaneously to 21 animals belonging to three groups A, B and C until all of them are dead. The following data give the lethal dose (in some suitable unit) of each animal at death.

Group	Lethal dose
A	84,47,34,41,60,45
B	40,108,117,95,86,59,98,67,61,92
C	90,93,100,46,93

The above data can be naturally ordered as

34	40	41	45	46	47	59	60	61	67	84	86	90	91	92	93	95	98	100	108	117
A	B	A	A	C	A	B	A	B	B	A	B	C	C	B	C	B	B	C	B	B

Denoting the data from group A as first population, from group B as second population and data from group C as third population we have

$n_1 = 6$ $\quad R_1 = 33$ $\quad N = 21$

$n_2 = 10$ $\quad R_2 = 131$

$n_3 = 5$ $\quad R_3 = 67$

Using (2.8), the Kruskal statistic $H(\equiv B_N^{(N)})$ can be computed as 6.61. By comparing the above value with that of ^{2} with 2 degrees of freedom we see that the H_o of equality of the three populations will be rejected.

Now, let $r = 14$. We can then compute from above

$n_{1r} = 6$ $\quad R_{1r} = 33$ $\quad NS_1 = -57$

$n_{2r} = 5$ $\quad R_{2r} = 40$ $\quad NS_2 = -35$

$n_{3r} = 3$ $\quad R_{3r} = 32$ $\quad NS_3 = -13$

So that, using (2.11) we find that $B_{14}^{(21)} = 7.05$ which also leads to the rejection of null hypothesis. Similarly computation with $r=9$ gives $B_9^{(21)} = 5.69$ which also leads to rejection of H_o. It is interesting to note that the usual one-way analysis of variance test gives the value of the F-ratio with (2,18) degrees of freedom as $F_{2,18} = 4.22$

which also leads to the rejection of H_o at the 5% level of significance.

Thus we have studied some properties of an "r out of N" test which seems to be suitable for testing the equality of k populations against location or scalar alternatives. In the next chapter we consider two other k-sample r-out of N tests which are suitable for a special type of alternative.

Chapter IV

OTHER k-SAMPLE EXTENSIONS WITH APPLICATIONS TO RANKING PROBLEMS

1. Introduction.

In this chapter we shall consider two k-sample extensions of the $V_r^{(N)}$ statistic for testing the null hypothesis H_o given by

(1.1) $$H_o: F_1(x) = F_2(x) = \cdots = F_k(x)$$

for all x against the ordered alternative

(1.2) $$H_1: F_1(x) < F_2(x) < \cdots < F_k(x)$$

for all x, where the $F_i(x)$ are labeled in such a way that (1.2) is the ordered alternative being considered. The above formulation is useful in that the rejection of the null hypothesis lends strong support to the possibility that the populations can be ordered (uniformly with respect to x) in a particular way. This formulation is useful, e.g. in life testing where the k cdf's might be associated with k different processes and the experimenter wishes to test whether the k processes give rise to units with the same life time distributions against the alternative that the processes can be ordered in a particular manner, in the sense that the k life-time distributions can be ordered uniformly with respect to x.

In section 2 we have defined the statistic V(N,r) and in section 3 we have derived the mean and the variance of V(N,r) under H_o. Section 4 is devoted in investigating the extreme values of V(N,r) and the asymptotic normality of V(N,r) under the H_o is proved in section 5. Finally in section 6 we have considered a second generalization of the $V_r^{(N)}$ statistic which generalizes the statistics proposed by Jonckheere [19] and Terpstra [31] .

2. Definition of the Statistic $V(N,r)$.

Let X_{ij} $(j=1,2,\cdots,n_i;\ i=1,2,\cdots,k)$ be k independent samples of sizes $n_1,n_2,\cdots,n_k$ from k-populations with continuous cdf's $F_1(x),F_2(x),\cdots,F_k(x)$ respectively. As before, let us assume that only the first r ordered observations out of the combined sample of size $N = \sum_{i=1}^{k} n_i$ are available. Let $n_{i\alpha}$ be the cumulative number of observations from the i^{th} population among the first α-ordered observations so that

$$\sum_{i=1}^{k} n_{i\alpha} = \alpha \qquad (\alpha=1,2,\cdots,r) \quad . \tag{2.1}$$

Define V_{ij} by

$$V_{ij} = \sum_{\alpha=1}^{r} (n_j n_{i\alpha} - n_i n_{j\alpha}) \qquad (i,j=1,2,\cdots,k;\ i<j) \quad . \tag{2.2}$$

Then the statistic $V(N,r)$ is defined by

$$V(N,r) = \sum_{i<j} V_{ij} \tag{2.3}$$

where the summation in (2.3) is over all pairs (i,j) with $i<j$. For $k=2$ it is clear that $V(N,r)$ is the same as the $V_r^{(N)}$ statistic proposed by Sobel [28] and defined in (1.3) of chapter II above.

3. Mean and Variance of $V(N,r)$ Under H_o.

In this section we shall find the mean and variance of $V(N,r)$ under H_o. To this end we define for each pair (i,j) with $i<j$ a sequence of random variables $z_{ij\alpha}$ $(\alpha=1,2,\cdots,r)$ by

$$z_{ij\alpha} = \begin{cases} +\,n_j \text{ if the } \alpha^{th} \text{ ordered observation is an } X_i \\ -\,n_i \text{ if the } \alpha^{th} \text{ ordered observation is an } X_j \\ 0, \text{ otherwise} \end{cases} \quad . \tag{3.1}$$

Lemma 3.1: For each given pair (i,j) the statistic V_{ij} can be expressed as a linear combination of the $z_{ij\alpha}$.

Proof. Letting $u_\beta = n_j n_{i\beta} - n_i n_{j\beta}$, we have from (2.2)

$$V_{ij} = \sum_{\beta=1}^{r} u_\beta = \sum_{\beta=1}^{r} \sum_{\alpha=1}^{\beta} z_{ij\alpha} \tag{3.2}$$

$$= \sum_{\alpha=1}^{r} (r+1-\alpha) z_{ij\alpha} \quad .$$

Using (3.1) and letting E_o denote expectation under H_o, we obtain by routine computation the following results which we state as

Theorem 3.1: Under H_o

$$E_o(z_{ij\alpha}) = 0 \tag{3.3}$$

$$E_o(z^2_{ij\alpha}) = n_i n_j (n_i + n_j)/N \tag{3.4}$$

$$E_o(z_{ij\alpha} z_{ij\alpha'}) = -\frac{n_i n_j (n_i + n_j)}{N(N-1)} \qquad (\alpha \neq \alpha') \tag{3.5}$$

$$E_o(z_{ij\alpha} z_{i'j'\alpha'}) = 0 \qquad (i \neq j', i' \neq j, i \neq i', j = j') \tag{3.6}$$

$$E_o(z_{ij\alpha} z_{ij'\alpha}) = n_i n_j n_j' /N \qquad (j \neq j') \tag{3.7}$$

$$E_o(z_{ij\alpha} z_{ij'\alpha'}) = -\frac{n_i n_j n_j'}{N(N-1)} \qquad (j \neq j', \alpha \neq \alpha') \tag{3.8}$$

$$E_o(z_{ij\alpha} z_{jt\alpha}) = -\frac{n_i n_j n_t}{N} \tag{3.9}$$

$$E_o(z_{ij\alpha} z_{jt\alpha'}) = \frac{n_i n_j n_t}{N(N-1)} \qquad (\alpha \neq \alpha') \tag{3.10}$$

$$E_o(z_{ij\alpha} z_{i'j\alpha}) = n_i n_{i'} n_j /N \qquad (i \neq i') \tag{3.11}$$

$$E_o(z_{ij\alpha} z_{i'j\alpha'}) = -\frac{n_i n_{i'} n_j}{N(N-1)} \qquad (i \neq i') \quad . \tag{3.12}$$

Let $\sigma_o^2(x)$ and $\sigma_o(x,y)$ denote respectively the variance of x and the covariance between x and y under the null hypothesis. We have

Theorem 3.2. Under H_o the mean and variance of V_{ij} are given by

(3.13) $$E_o(V_{ij}) = 0,$$

and

(3.14) $$\sigma_o^2(V_{ij}) = \frac{n_i n_j(n_i+n_j)r(r+1)[2N(2r+1)-3r(r+1)]}{12\,N(N-1)} = \frac{n_i n_j(n_i+n_j)N^2}{G}$$

where G has been defined by (2.4) of chapter III.

Proof. Using (3.2, (3.3), and (3.4) we obtain

$$E_o(V_{ij}) = \sum_{\alpha=1}^{r} (r+1-\alpha)E_o(z_{ij\alpha}) = 0$$

and

$$\sigma_o^2(V_{ij}) = \sum_{\alpha=1}^{r} (r+1-\alpha)^2\,\sigma_o^2(z_{ij\alpha}) + \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{r} (r+1-\alpha)(r+i-\beta)\,\sigma_o(z_{ij\alpha}, z_{ij\beta})$$

$$= \frac{n_i n_j(n_i+n_j)}{N}\left[\sum_{\alpha=1}^{r}(r+1-\alpha)^2 - \frac{\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{r}(r+1-\alpha)(r+1-\beta)}{N-1}\right]$$

$$= \frac{n_i n_j(n_i+n_j)r(r+1)\,[2N(2r+1)-3r(r+1)]}{12\,N(N-1)} .$$

From theorem 3.2 we have

Corollary 3.1. $E_o(V(N,r)) = 0$.

The proof follows from the definition of $V(N,r)$.

To find the variance of $V(N,r)$ under H_o we need a few more results.

Let

$$n_{(12\ldots j)} = \sum_{i=1}^{j} n_i \quad\text{and}\quad n_{(12\ldots j)\alpha} = \sum_{i=1}^{j} n_{i\alpha} .$$

Define $V_{(12\ldots j)t}$ for $j<t$ by

(3.15) $$V_{(12\ldots j)t} = \sum_{\alpha=1}^{r} (n_t n_{(12\ldots j)\alpha} - n_{(12\ldots j)}n_{t\alpha}) .$$

i.e., $V_{(12\cdots j)t}$ is the usual V_{ij} statistic as defined in (2.2) where the first j groups have been pooled together to form one sample. From the definition (3.15) it is clear that

$$(3.16)\qquad V_{(12\cdots j)t} = \sum_{i=1}^{j} V_{it} \quad .$$

We can now prove

Lemma 3.2:

$$(3.17)\qquad V(N,r) = \sum_{j=2}^{k} V_{(12\cdots,j-1)j} \quad ,$$

$$(3.18)\qquad E_o\{V_{(12\cdots,j-1)j}\} = 0 \quad ,$$

$$(3.19)\qquad \sigma_o^2\{V_{(12\cdots,j-1)j}\} = n_{(12\cdots,j-1)}n_j\, n_{(12\cdots j)}N^2/G,$$

and

$$(3.20)\qquad \sigma_o\{V_{(12\cdots,j-1)j},\ V_{(12\cdots,j'-1)j'}\} = 0 \qquad (j\neq j') \quad .$$

Proof. Using (2.3) and (3.14) we readily prove (3.17), since

$$V(N,r) = \sum_{i=1}^{k-1}\ \sum_{j=i+1}^{k} V_{ij} = \sum_{j=2}^{k}\left(\sum_{i=1}^{j-1} V_{ij}\right) = \sum_{j=2}^{k} V_{(12\cdots,j-1)j} \quad .$$

Clearly, (3.18) is obvious as

$$E_o\{V_{(12\cdots,j-1)j}\} = \sum_{i=1}^{j-1} E_o(V_{ij}) = 0 \quad .$$

Now

$$(3.21)\qquad \sigma_o^2\{V_{(12\cdots,j-1)j}\} = \sum_{i=1}^{j-1} \sigma_o^2(V_{ij}) + \sum_{i\neq i'=1}^{j-1} \sigma_o(V_{ij},V_{i'j})$$

From (3.11) and (3.12) we obtain

$$(3.22)\qquad \sigma_o(V_{ij},V_{i'j}) = \frac{n_i n_{i'} n_j}{N}\left[\sum_{\alpha=1}^{r}(r+1-\alpha)^2 - \frac{\sum_{\alpha\neq\beta'=1}^{r}(r+1-\alpha)(r+1-\beta)}{N-1}\right]$$

$$= \frac{n_i n_{i'} n_j N^2}{G} \quad .$$

Using (3.14) and (3.22) in (3.21), we obtain

$$\sigma_o^2 \{V_{(12\ldots,j-1)j}\} = \frac{n_j N^2}{G} \left[\sum_{i=1}^{j-1} n_i(n_i+n_j) + \sum_{i\neq i'=1}^{j-1} n_i n_{i'} \right]$$

$$= n_{(12\ldots,j-1)} n_j n_{(12\ldots j)} N^2/G,$$

which proves (3.19).

Finally to prove (3.20) we assume without any loss of generality that $j < j'$. From (3.2) and (3.6) we obtain

$$(3.23) \qquad E(V_{ij}V_{i'j'}) = 0 \text{ for } i\neq i', j\neq j', j\neq i' \text{ and } i\neq j' .$$

Hence, we have, using (3.2), (3.7), (3.8), (3.9) and (3.10)

$$\sigma_o^2 \{V_{(12\ldots,j-1)j} V_{(12\ldots,j'-1)j'}\} = E_0 \sum_{i=1}^{j-1} (V_{ij}V_{ij'} + V_{ij}V_{jj'})$$

$$= \sum_{i=1}^{j-1} \left[\frac{n_i n_j n_{j'}}{N} \left\{ \sum_{\alpha=1}^{r} (r+1-\alpha)^2 - \sum_{\alpha\neq\alpha'=1}^{r} (r+1-\alpha)(r+1-\alpha') \right\} + \frac{n_i n_j n_{j'}}{N} \left\{ - \sum_{\alpha=1}^{r} (r+1-\alpha)^2 + \frac{\sum_{\alpha\neq\alpha'=1}^{r} (r+1-\alpha)(r+1-\alpha')}{N-1} \right\} \right]$$

$$= 0 .$$

The variance of $V(N,r)$ under H_o now follows directly from lemma 3.2 and is given as

<u>Theorem 3.3:</u> Under H_o the variance of $V(N,r)$ is given by

$$(3.24) \qquad \sigma_o^2(V(N,r)) = \frac{N^2}{G} \sum_{j=2}^{k} \{n_j n_{(12\ldots,j-1)} n_{(12\ldots j)}\} \quad .$$

4. <u>Extreme Values of V(N,r)</u>.

It is of interest to know the extreme values of $V(N,r)$. From the definition of $V_{(12\ldots j)}$ it can be seen that the values of $V_{(12\ldots j)t}$ for $t \geq j+1$ remain unchanged for any permutation of the observations corresponding to different cdf's F_1 through F_j. Thus to find the maximum value of $V(N,r)$ we shall find the permutation of the observations belonging to the first j samples (keeping all observations from other samples fixed) for which $V_{(12\ldots,j-1)j}$ is maximum. Afterwards, we will permute between the (j+1)st and the pooled set consisting of the first j samples so that $V_{(12\ldots j)j+1}$ is a maximum. Continuing this process we find the maximum values for each $V_{(12\ldots,j-1)j}$ $(j=2,3,\ldots,k)$. From (3.17) the maximum of $V(N,r)$ follows. Similarly, we can also find the minimum of $V(N,r)$.

From (2.2) it is clear that V_{12} is maximum if the observations from the first sample precede the observations from the second sample (among the first r observations). Similarly, V_{12} and $V_{(12)3}$ are simultaneously maximum if the observations, from the first sample precede those from the second sample and observations from the second sample precede the observations from the third sample, (while observations from all other samples are kept fixed). Proceeding in this way we see that all the $V_{(12\ldots,j-1)j}$ are simultaneously maximum (and therefore $V(N,r)$ is maximum) when the first r_1 observations are from the first population, the next r_2 observations are from the second population and so forth where

$$(4.1) \qquad r_i = \begin{cases} \min(n_i, r-n_1-n_2-\cdots-n_{i-1}) & \text{if } r \geq n_1 \\ 0 \quad \text{otherwise} & (i=1,2,\ldots,k). \end{cases}$$

Below we compute for k=3 the maximum values of $V(N,r)$ for different possible values of r.

For $r \leq n_1$, $V_{12} = \dfrac{rn_2(r+1)}{2}$, $V_{(12)3} = \dfrac{rn_3(r+1)}{2}$ and

$$V(N,r) = (n_2+n_3)r(r+1)/2 \quad . \tag{4.2}$$

for $n_1 < r \leq n_1+n_2$, $V_{12} = \frac{1}{2}\,[n_1n_2(2r-n_1+1)-n_1(r-n_1)(r-n_1+1)]$,
$V_{(12)3} = rn_3(r+1)/2$, and

$$V(N,r) = \tfrac{1}{2}\,[n_1(r-n_1+1)(n_1+n_2-r)+r\,\{n_1n_2+n_3(r+1)\}] \quad . \tag{4.3}$$

Finally, for $n_1+n_2 < r \leq n_1+n_2+n_3$

$$V_{12} = \frac{n_1n_2(n_1+n_2)}{2}\ ,\quad V_{(12)3} = \frac{(n_1+n_2)}{2}\,[(N-r)(r-n_1-n_2+1)+rn_3]$$

and

$$V(N,r) = \frac{(n_1+n_2)}{2}\,[(n_1n_2+rn_3)+(N-r)(r-n_1-n_2+1)] \quad . \tag{4.4}$$

Similarly we can compute the minimum value of $V(N,r)$ where the order in which the observations occur is exactly the reverse of the case for which $V(N,r)$ is maximum. That is, observations from k-th population will precede all other observations, observations of (k-1)-st population will precede all but the observations from the k-th population and so forth. In the special case $k=3$, we compute the minimum values of $V(N,r)$ below. For $r \leq n_3$

$$V(N,r) = -(n_1+n_2)r(r+1)/2 \quad . \tag{4.5}$$

For $n_3 < r \leq n_2+n_3$

$$V(N,r) = \tfrac{1}{2}\,[n_3(r-n_3+1)(r-n_2-n_3)-r\,\{n_2n_3+n_1(r+1)\}] \quad , \tag{4.6}$$

and for $n_2+n_3 < r \leq n_1+n_2+n_3$

$$V(N,r) = \tfrac{1}{2}\,[n_2(r-n_2-n_3+1)(r-N)+n_3(r-n_3+1)(r-N) - n_1n_2(r-n_3)-(n_1+n_2)n_3r] \quad . \tag{4.7}$$

It can be easily seen from the above equation that in each case if $(n_1,n_2,n_3) = (n_3,n_2,n_1)$ then

(4.8) maximum value of $V(N,r)$ + minimum value of $V(N,r) = 0$.

Also if $n_1=n_2=n_3$ then the minimum difference between two successive values of $V(N,r)$ will be $2n$ since V_{12} changes by steps of $2n$ when all other $V_{(12\cdots,j-1)j}$ are kept fixed.

5. Asymptotic Normality of $V(N,r)$.

In this section we shall prove the asymptotic normality of $V(N,r)$ under H_o as $N,n_i \to \infty$ with $n_i/N \to \lambda_i > 0$. To this end let us define the random variables $\delta^*_{i\alpha}$ by

$$(5.1) \qquad \delta^{*}_{i\alpha} = \begin{cases} \alpha \text{ if } (r+1-\alpha)\text{-th ordered observation is an } x_i \\ 0 \quad \text{otherwise} \end{cases}$$

that is,

$$(5.2) \qquad \delta^*_{i\alpha} = \alpha z^{(i)}_{r+1-\alpha} \qquad (i=1,2,\cdots,k),$$

where $z^{(i)}_{r+1-\alpha}$ has been defined in equation (2.1) of chapter III.

We have

Lemma 5.1

$$(5.3) \qquad \sum_{\alpha=1}^{r} n_{i\alpha} = \sum_{\alpha=1}^{r} \delta^*_{i\alpha} \qquad (i=1,2,\cdots,k) \quad .$$

Proof. The proof of this lemma is exactly the same as that of lemma 2.1 of chapter II.

Using lemma 5.1 we now prove

Theorem 5.1. Each V_{ij} is asymptotically normally distributed under H_o.

Proof. From (2.2), (5.2) and (5.3)

$$V_{ij} = n_j \sum_{\alpha=1}^{r} \delta^*_{i\alpha} - n_i \sum_{\alpha=1}^{r} \delta^*_{j\alpha}$$

$$= n_j \sum_{\alpha=1}^{r} (r+1-\alpha) z_\alpha^{(i)} - n_i \sum_{\alpha=1}^{r} (r+1-\alpha) z_\alpha^{(i)}$$

$$= N\, n_i n_j (T_{N,j} - T_{N,i})$$

where the $T_{N,i}$'s have been defined in (4.13) of chapter III and have been shown to have a limiting (k-1)-dimensional normal distribution. It follows that the V_{ij}'s are asymptotically normally distributed under the null as well as the non-null hypothesis.

Since $V(N,r)$ is a linear combination of the V_{ij}'s from theorem 3.1 the asymptotic normality of $V(N,r)$ follows.

6. A Second k-sample Extension.

In this section we shall consider another k-sample extension of the Wilcoxon statistic which generalizes the statistic proposed by Jonckheere [19] and Terpstra [31] . As in the above the proposed statistic $W(N,r)$ is an r out of N statistic. However, the results of this section will be conditioned on a given pattern $n_{1r}, n_{2r}, \cdots, n_{kr}$ (and of course their sum r is fixed also) so that the type of censoring considered is the same as that considered by Halperin [18] and Gehan [15] .

The statistic $W(N,r)$ is based on the quantities w_{ij} which are computed from each pair (i, j) of samples as shown below. Let us order the $n_{ir} + n_{jr} = n_{(ij)r}$ observations from the i-th and j-th sample among themselves and let $n_{i\alpha}^{(ij)}$ be the cumulative number of failures from the i-th population among the first α-ordered observations so that

$$n_{i\alpha}^{(ij)} + n_{j\alpha}^{(ij)} = \alpha \qquad (\alpha=1,2,\cdots,n_{(ij)r}) \quad . \tag{6.1}$$

The quantity w_{ij} is then defined by

$$w_{ij} = \sum_{\alpha=1}^{n_{(ij)r}} (n_{jr}n_{i\alpha}^{(ij)} - n_{ir}n_{j\alpha}^{(ij)}) \qquad ((i,j)=1,2,\cdots,k;i< j) \tag{6.2}$$

and the statistic $W(N,r)$ is defined as

$$W(N,r) = \sum_{i<j} \frac{w_{ij}}{n_{(ij)r}} \tag{6.3}$$

where the summation in (6.3) is over all pairs (i,j) with $i<j$.

The statistic $W(N,r)$ is related to the $V_r^{(N)}$ statistic defined in (1.3) of chapter II. For, when $k=2$ and $r=N$ we have $n_{(12)r} = N$ and $NW(N,N) = w_{12} = V_N^{(N)}$. Also for $r=N$, each w_{ij} is a $V_r^{(N)}$ statistic based on the i-th and j-th sample only. Hence, from (4.9) of Sobel [29] we have in the above case

$$\frac{w_{ij}}{n_i+n_j} = U_{ij} - \frac{n_in_j}{2} \tag{6.4}$$

where U_{ij} is the number of pairs $(X_{i\alpha},X_{j\beta})$ with $X_{i\alpha}< X_{j\beta}$ summed over all possible pairs (α,β). Using (6.3) and (6.4) we obtain

$$W(N,N) = \sum_{i<j} (U_{ij} - \frac{n_in_j}{2}) = S/2. \tag{6.5}$$

where $S = \sum_{i<j} (2U_{ij}-n_in_j)$ is the statistic proposed by Jonckheere [19] and is related to the Terpstra statistic [31]. In this sense $W(N,r)$ for $r\leq N$ can be considered as a generalization of the S statistic.

Since the statistic S is asymptotically normally distributed so is $W(N,N)$. Now, for fixed n_{ir} and n_{jr} the statistic w_{ij} is of the same form as the $V_r^{(N)}$ statistic. Thus, as $N,r,n_1,n_2,\cdots,n_k$, $n_{1r},\cdots,n_{kr}$ all tend to infinity with

$$\lim_{N \to \infty} \frac{r}{N} = p, \quad \lim_{N \to \infty} \frac{n_i}{N} = \lambda_i > 0, \quad \lim_{N \to \infty} \frac{n_{ir}}{r} = \lambda_{ir} > 0 \quad (i=1,2,\cdots,k)$$

each $\frac{w_{ij}}{n_{(ij)r}}$ is asymptotically normally distributed. Hence the asymptotic distribution of $W(N,r)$ both under the null and the non-null hypotheses is also normal.

Appendix I

Expected Value of $(B_r^{(N)})^2$ Under H_o

In this appendix we compute $E_o(B_r^{(N)})^2$ under the null hypothesis. From (2.11) of chapter III, we have

$$(A.1)\quad (B_r^{(N)})^2 = G^2\Big(\sum_{i=1}^{k}\frac{S_i^4}{n_i^2} + \sum_{\substack{i,j=1\\ i\neq j}}^{k}\frac{S_i^2S_j^2}{n_in_j} - \frac{r^2(r+1)^2}{2N^3}\sum_{i=1}^{k}\frac{S_i^2}{n_i} + \frac{r^4(r+1)^4}{16N^6}\Big) \quad .$$

For further computation we shall make use of the following elementary results where all the summations are from $\alpha=1$ to $\alpha=r$ and within any term of a summation involving several indices (α,β , γ and δ) no two of which are equal.

(a) $$\Sigma\alpha = -\Sigma(\alpha-r-1) = \frac{r(r+1)}{2}$$

$$\Sigma\alpha^2 = \Sigma(\alpha-r-1)^2 = \frac{r(r+1)(2r+1)}{6}$$

$$\Sigma\alpha^3 = -\Sigma(\alpha-r-1)^3 = \left(\frac{r(r+1)}{2}\right)^2$$

$$\Sigma\alpha^4 = \Sigma(r-\alpha-1)^4 = \frac{r(r+1)(2r+1)(3r^2+3r-1)}{30}$$

(b) $$(\Sigma\alpha)^2 = \Sigma\alpha^2 + \Sigma\alpha\beta$$

$$(\Sigma\alpha)^3 = \Sigma\alpha^3 + 3\Sigma\alpha^2\beta + \Sigma\alpha\beta\gamma$$

$$(\Sigma\alpha)^4 = \Sigma\alpha^4 + 4\Sigma\alpha^3\beta + 3\Sigma\alpha^2\beta^2 + 6\Sigma\alpha^2\beta\gamma + \Sigma\alpha\beta\gamma\delta$$

(c) $$\Sigma\alpha^2\beta\gamma = (\Sigma\alpha^2)(\Sigma\alpha)^2 - 2(\Sigma\alpha^3)(\Sigma\alpha) - (\Sigma\alpha^2)^2 + 2\Sigma\alpha^4$$

(d) $$\Sigma\alpha\beta\gamma\delta = (\Sigma\alpha)^4 + 8(\Sigma\alpha^3)(\Sigma\alpha) + 3(\Sigma\alpha^2)^2 - 6(\Sigma\alpha^2)(\Sigma\alpha)^2 - 6\Sigma\alpha^4.$$

Using these results and theorem 3.1 we get

(A.2) $$E_o(S_i^{\,4}) = E_o\left[\sum_{\alpha=1}^{r}\left(\frac{\alpha-r-1}{N}\right)z_\alpha^{(i)}\right]^4$$

$$= \frac{(r+1)^{(2)}(2r+1)(3r^2+3r-1)n_i}{30N^5} + \frac{(r+1)^{(3)}(50r^3+66r^2-5r-14)n_i^{(2)}}{60N^4\cdot N^{(2)}}$$

$$+ \frac{(r+1)^{(4)}(30r^3+35r^2-11r-12)n_i^{(3)}}{60N^4\cdot N^{(3)}}$$

$$+ \frac{(r+1)^{(5)}(15r^3+15r^2-10r-8)n_i^{(4)}}{240N^4\cdot N^{(4)}}$$

where $x^{(t)} = x(x-1)(x-2)\cdots(x-t+1)$.

Hence

(A.3) $$N^4\sum_{i=1}^{k}\frac{E_o(S_i^4)}{n_i^2} = \frac{(r+1)^{(2)}(2r+1)(3r^2+3r-1)}{30N}\sum_{i=1}^{k}\frac{1}{n_i}$$

$$+ \frac{(r+1)^{(3)}(50r^3+66r^2-5r-14)}{60N^{(2)}}\left(k - \sum_{i=1}^{k}\frac{1}{n_i}\right)$$

$$+ \frac{(r+1)^{(4)}(30r^3+35r^2-11r-12)n_i^{(3)}}{60N^{(3)}}\left(N-3k+2\sum_{i=1}^{k}\frac{1}{n_i}\right)$$

$$+ \frac{(r+1)^{(5)}(15r^3+15r^2-10r-8)}{240N^{(4)}}\left(\sum_{i=1}^{k}n_i^{\,2}-6N+11k-6\sum_{i=1}^{k}\frac{1}{n_i}\right) .$$

Similarly, with c_α defined by (3.8) of chapter III, we have

(A.4) $$E_o(S_i^2S_j^2) = \Sigma\, c_\alpha c_{\alpha'} c_\beta c_{\beta'} E_o(z_\alpha^{(i)}z_{\alpha'}^{(i)}z_\beta^{(j)}z_{\beta'}^{(j)})$$

$$+ \{\Sigma\, c_\alpha c_{\alpha'} c_\beta^2 E_o(z_\alpha^{(i)}z_{\alpha'}^{(i)}z_\beta^{2(j)}) + \Sigma\, c_\alpha^2 c_\beta c_{\beta'} E_o(z_\alpha^{2(i)}z_\beta^{(j)}z_{\beta'}^{(j)})\}$$

$$+ \Sigma\, c_\alpha^2 c_\beta^2 E_o(z_\alpha^{2(i)}z_\beta^{2(j)})$$

$$= \frac{(r+1)^{(5)}(15r^3+15r^2-10r-8)}{240N^4}\cdot\frac{n_in_j(n_i-1)(n_j-1)}{N^{(4)}}$$

$$+\frac{(r+1)^{(4)}(30r^3+35r^2-11r-12)}{360N^4}\cdot\frac{n_in_j(n_i+n_j-2)}{N^{(3)}}$$

$$+\frac{(r+1)^{(3)}(2r+1)(10r^2+7r-6)}{180N^4}\cdot\frac{n_in_j}{N^{(2)}}\quad.$$

Hence

$$\text{(A.5)}\quad N^4\sum_{\substack{i,j=1\\ i\neq j}}^{k}\frac{E_o(S_i^2S_j^2)}{n_in_j}=\frac{(r+1)^{(5)}(15r^3+15r^2-10r-8)}{240N^{(4)}}\left\{\begin{array}{l}N^2-\sum_{i=1}^{k}n_i^2-2N(k-1)\\ \quad+k(k-1)\end{array}\right\}$$

$$+\frac{(r+1)^{(4)}(30r^3+35r^2-11r-12)}{180N^{(3)}}\quad\{N(k-1)-(k-1)k\}$$

$$+\frac{(r+1)^{(3)}(20r^3+24r^2-5r-6)}{180N^{(2)}}\,k(k-1)$$

Finally,

$$\text{(A.6)}\qquad\sum_{i=1}^{k}\frac{E_o(S_i^2)}{n_i}=\frac{r(r+1)(2r+1)}{6N^3}\,k+\frac{r^{(2)}(r+1)(3r+2)}{12N^2\cdot N^{(2)}}(N-k)$$

From (A.1), (A.3), (A.5) and (A.6) we obtain

$$\text{(A.7)}\qquad\frac{N^4E_o(B_r^{(N)})^2}{G^2}=\frac{(r+1)^{(2)}(6r^3+9r^2+r-1)}{30N}\sum_{i=1}^{k}\frac{1}{n_i}$$

$$+\frac{(r+1)^{(5)}(15r^3+15r^2-10r-8)}{240N^{(3)}}(N^2-2Nk-4N+k^2+10k-6\sum_{i=1}^{k}\frac{1}{n_i})$$

$$+\frac{(r+1^{(4)}(30r^3+35r^2-11r-12)}{180N^{(3)}}(Nk+2N-k^2-8k+6\sum_{i=1}^{k}\frac{1}{n_i})$$

$$+\frac{(r+1)^{(3)}}{180N^{(2)}}\quad\begin{array}{l}\{(20r^3+24r^2-5r-6)k^2+(130r^3+174r^2-10r-36)k\\ \quad-(150r^3+198r^2-15r-42)\sum_{i=1}^{k}\frac{1}{n_i}\ \}\end{array}$$

$$-\frac{r^3(r+1)^3}{48N^2(N-1)}\quad\{3Nr^2+3r^2-5Nr+3r-4N+(8Nr-6r^2+4N-6r)k\ \}\quad.$$

TABLE I

ARE of $T_r^{(N)}$ Test with Respect to the Precedence

Test for Different Densities and

for $\delta = \frac{1}{2}$ and $p = 1$.

$f(x,\theta)$	$\sigma^2=[4f^2(\eta_{\frac{1}{2}},\theta_o)]^{-1}$	$[\int_{-\infty}^{\infty} f^2(x,\theta_o)dx]^2$	$e(T_r^{(N)},P)$
$\frac{1}{2\pi} e^{-(x-\theta)^2/2}$, $-\infty < x < \infty$	$\pi/2$	$1/4\pi$	1.50
1, $\theta-\frac{1}{2} \leq x \leq \theta+\frac{1}{2}$	$1/4$	1	3.00
$\theta e^{-\theta x}$, $\theta_o=1, 0 < x < \infty$	1	$1/4$	6.24
$\theta t e^{-\theta x^t} x^{t-1}$, $\theta_o=1, 0<x<\infty$	$\frac{1}{t^2(\ln 2)^{2-2/t}}$	$\frac{t^2\Gamma^2(2-1/t)}{2^{4-2/t}}$	$\frac{6.24t^2\Gamma^2(2-1/t)}{2^{2-2/t}}$
$\frac{\theta^t}{t} e^{-\theta x} x^{t-1}$, $\theta_o=1, 0<x<\infty$		$\frac{t^2\Gamma^2(2-1/t)}{2^{4-2/t}}$	*

* can not be computed in explicit form

TABLE II

ARE of Different Statistics with Respect to Sobel Statistic $T_r^{(N)}$ in the Case of the Exponential Distribution for Different Values of p

p	$e(T_r^{(N)},F)$ $= e(T_r^{(N)},S_r^{(N)})$	$e(T_r^{(N)},G^{(N)})$	$e(T_r^{(N)},P)$ $\delta = 1-e^{-1}$
0		.75	
.1	.7550	.78	.1297
.2	.7627	.81	.2621
.4	.7753	.88	.5328
.5	.7812	.91	.6711
.6	.7864	.94	.8107
.75	.7911	.9796	1.0195
.8	.7908	.9879	1.0870
.9	.7835	.9979	1.2117
1.0	.7500	1.0000	1.2886

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